Active Calculus - Multivariable - Activities Workbook

2015 edition

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Chapter 9

Multivariable and Vector Functions

9.1 Functions of Several Variables and Three Dimensional Space

Preview Activity 9.1. When people buy a large ticket item like a car or a house, they often take out a loan to make the purchase. The loan is paid back in monthly installments until the entire amount of the loan, plus interest, is paid. The monthly payment that the borrower has to make depends on the amount P of money borrowed (called the principal), the duration t of the loan in years, and the interest rate r. For example, if we borrow $18,000 to buy a car, the monthly payment M that we need to make to pay off the loan is given by the formula

$$M = \frac{1500r}{1 - \frac{1}{\left(1+\frac{r}{12}\right)^{12t}}}.$$

The variables r and t are independent of each other, so using functional notation we write

$$M(r,t) = \frac{1500r}{1 - \frac{1}{\left(1+\frac{r}{12}\right)^{12t}}}.$$

(a) Find the monthly payments on this loan if the interest rate is 6% and the duration of the loan is 5 years.

(b) Evaluate $M(0.05, 4)$. Explain in words what this calculation represents.

(c) Now consider only loans where the interest rate is 5%. Calculate the monthly payments as indicated in Table 9.1. Round payments to the nearest penny.

(d) Now consider only loans where the duration is 3 years. Calculate the monthly payments as indicated in Table 9.2. Round payments to the nearest penny.

(e) Describe as best you can the combinations of interest rates and durations of loans that result in a monthly payment of $200.

Duration (in years)	2	3	4	5	6
Monthly payments (dollars)					

Table 9.1: Monthly payments at an interest rate of 5%.

Interest rate	0.03	0.05	0.07	0.09	0.11
Monthly payments (dollars)					

Table 9.2: Monthly payments over three years.

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Activity 9.1.

Identify the domain of each of the following functions. Draw a picture of each domain in the x-y plane.

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $Q(x, y) = \frac{x+y}{x^2-y^2}$

(d) $s(x, y) = \frac{1}{\sqrt{1-xy^2}}$

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$x \backslash y$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
25	7.6	14.0	18.2	19.5	17.8	13.2	6.5
50	30.4	56.0	72.8	78.1	71.0	52.8	26.2
75	68.4	126.1	163.8	175.7	159.8	118.7	58.9
100	121.7	224.2	291.3	312.4	284.2	211.1	104.7
125	190.1	350.3	455.1	488.1	444.0	329.8	163.6
150	273.8	504.4	655.3	702.8	639.3	474.9	235.5
175	372.7	686.5	892.0	956.6	870.2	646.4	320.6
200	486.8	896.7	1165.0	1249.5	1136.6	844.3	418.7
225	616.2	1134.9	1474.5	1581.4	1438.5	1068.6	530.0
250							

Table 9.3: Values of $f(x, y) = \frac{x^2 \sin(2y)}{g}$.

Activity 9.2.

Complete the last row in Table 9.3 to provide the needed values of the function f.

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Activity 9.3.

(a) Consider the set of points (x, y, z) that satisfy the equation $x = 2$. Describe this set as best you can.

(b) Consider the set of points (x, y, z) that satisfy the equation $y = -1$. Describe this set as best you can.

(c) Consider the set of points (x, y, z) that satisfy the equation $z = 0$. Describe this set as best you can.

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Activity 9.4.

Let $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ be two points in $\mathbb{R}^3$. These two points form opposite vertices of a rectangular box whose sides are planes parallel to the coordinate planes as illustrated in Figure 9.1, and the distance between P and Q is the length of the diagonal shown in Figure 9.1.

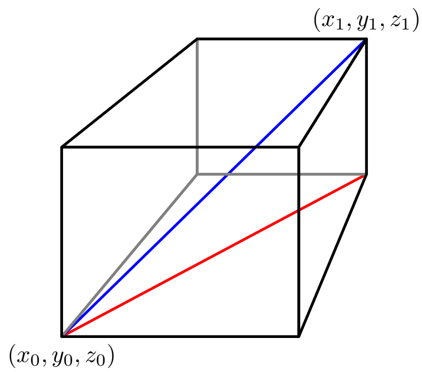

Figure 9.1: The distance formula in $\mathbb{R}^3$.

(a) Consider one of the right triangles in the base of the box whose hypotenuse is shown as the red line in Figure 9.1. What are the vertices of this triangle? Since this right triangle lies in a plane, we can use the Pythagorean Theorem to find a formula for the length of the hypotenuse of this triangle. Find such a formula, which will be in terms of x_0, y_0, x_1, and y_1.

(b) Now notice that the triangle whose hypotenuse is the blue segment connecting the points P and Q with a leg as the hypotenuse of the triangle found in part (a) lies entirely in a plane, so we can again use the Pythagorean Theorem to find the length of its hypotenuse. Explain why the length of this hypotenuse, which is the distance between the points P and Q, is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

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Activity 9.5.

In the following questions, we investigate the use of traces to better understand a function through both tables and graphs.

(a) Identify the $y = 0.6$ trace for the range function $f(x, y) = \frac{x^2 \sin(2y)}{g}$ by highlighting or circling the appropriate cells in Table 9.3. Write a sentence to describe the behavior of the function along this trace.

(b) Identify the $x = 150$ trace for the range function by highlighting or circling the appropriate cells in Table 9.3. Write a sentence to describe the behavior of the function along this trace.

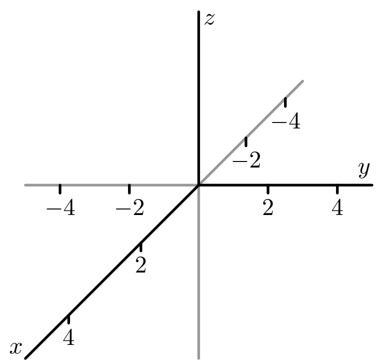

Figure 9.2: Coordinate axes to sketch traces.

(c) For the function $g(x, y) = x^2 + y^2 + 1$, explain the type of function that each trace in the x direction will be (keeping y constant). Plot the $y = -4$, $y = -2$, $y = 0$, $y = 2$, and $y = 4$ traces in 3-dimensional coordinate system provided in Figure 9.2.

(d) For the function $g(x, y) = x^2 + y^2 + 1$, explain the type of function that each trace in the y direction will be (keeping x constant). Plot the $x = -4$, $x = -2$, $x = 0$, $x = 2$, and $x = 4$ traces in 3-dimensional coordinate system in Figure 9.2.

(e) Describe the surface generated by the function g.

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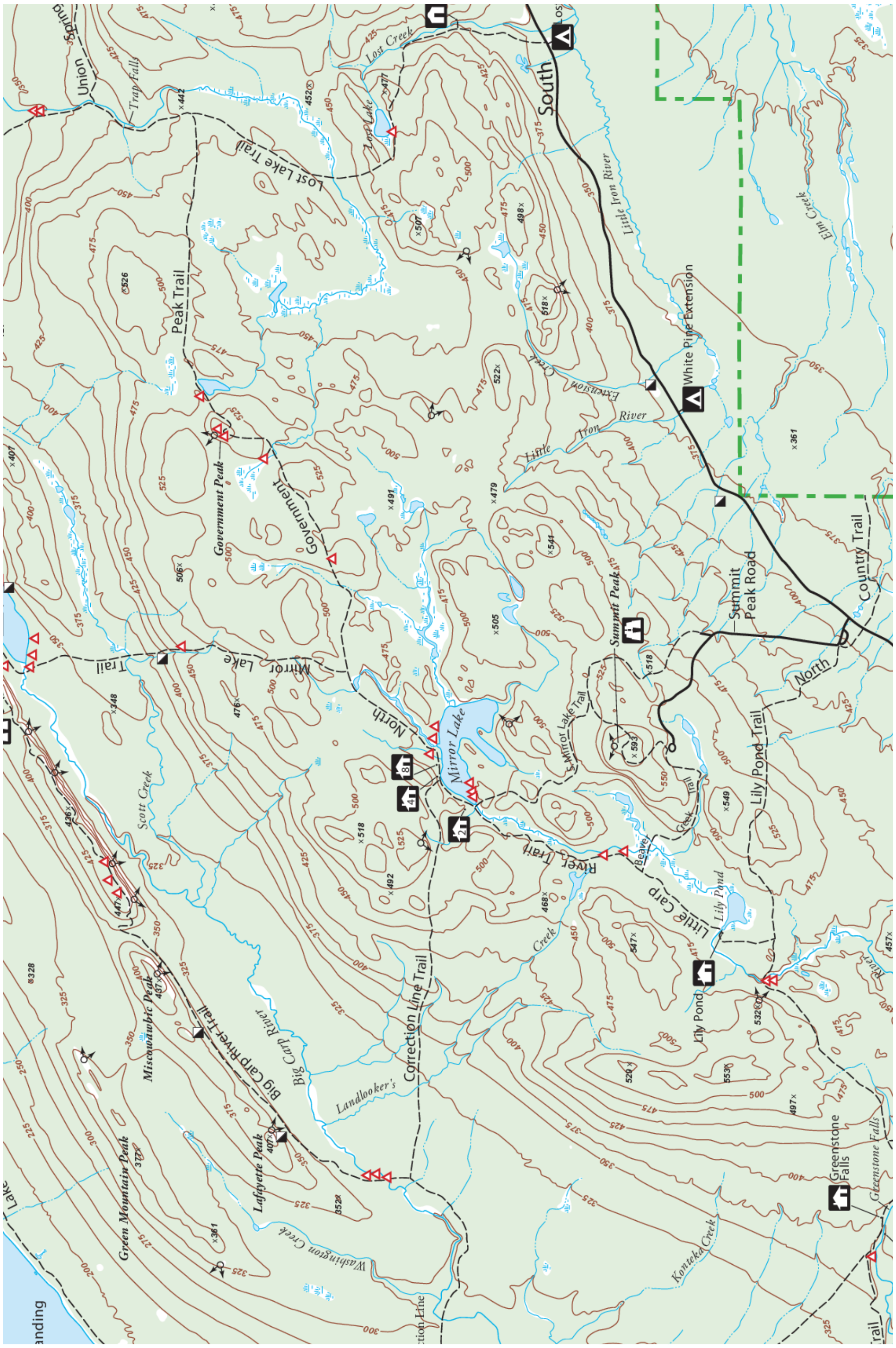

Figure 9.3: Contour map of the Porcupine Mountains.

Activity 9.6.

On the topographical map of the Porcupine Mountains in Figure 9.3,

(a) identify the highest and lowest points you can find;

(b) from a point of your choice, determine a path of steepest ascent that leads to the highest point;

(c) from that same initial point, determine the least steep path that leads to the highest point.

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Activity 9.7.

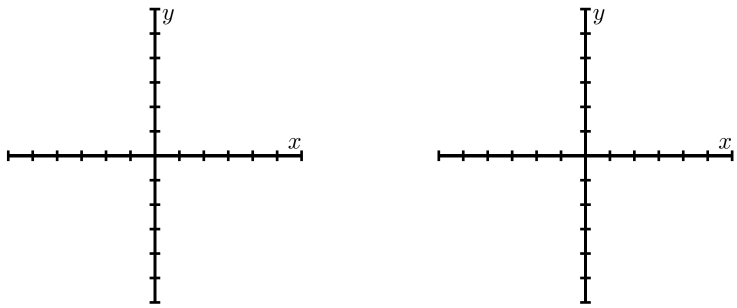

Figure 9.4: Left: Level curves for $f(x, y) = x^2 + y^2$. Right: Level curves for $g(x, y) = \sqrt{x^2 + y^2}$.

(a) Let $f(x, y) = x^2 + y^2$. Draw the level curves $f(x, y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the left set of axes given in Figure 9.4. (You decide on the scale of the axes.) Explain what the surface defined by f looks like.

(b) Let $g(x, y) = \sqrt{x^2 + y^2}$. Draw the level curves $g(x, y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the right set of axes given in Figure 9.4. (You decide on the scale of the axes.) Explain what the surface defined by g looks like.

(c) Compare and contrast the graphs of f and g. How are they alike? How are they different? Use traces for each function to help answer these questions.

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9.2 Vectors

Preview Activity 9.2. After working out, Sarah and John leave the Recreation Center on the Grand Valley State University Allendale campus (a map of which is given in Figure **??**) to go to their next classes.[1] Suppose we record Sarah's movement on the map in a pair $\langle x, y \rangle$ (we will call this pair a *vector*), where x is the horizontal distance (in feet) she moves (with east as the positive direction) and y as the vertical distance (in feet) she moves (with north as the positive direction). We do the same for John. Throughout, use the legend to estimate your responses as best you can.

(a) What is the vector $\mathbf{v}_1 = \langle x, y \rangle$ that describes Sarah's movement if she walks directly in a straight line path from the Recreation Center to the entrance at the northwest end of Mackinac Hall? (Assume a straight line path, even if there are buildings in the way.) Explain how you found this vector. What is the total distance in feet between the Recreation Center and the entrance to Mackinac Hall? Measure the number of feet directly and then explain how to calculate this distance in terms of x and y.

(b) What is the vector $\mathbf{v}_2 = \langle x, y \rangle$ that describes John's change in position if he walks directly from the Recreation Center to Au Sable Hall? How many feet are there between Recreation Center to Au Sable Hall in terms of x and y?

(c) What is the vector $\mathbf{v}_3 = \langle x, y \rangle$ that describes the change in position if John walks directly from Au Sable Hall to the northwest entrance of Mackinac Hall to meet up with Sarah after class? What relationship do you see among the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? Explain why this relationship should hold.

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[1] GVSU campus map from http://www.gvsu.edu/homepage/files/pdf/maps/allendale.pdf, used with permission from GVSU, credit to illustrator Chris Bessert.

Activity 9.8.

As a class, determine a coordinatization of your classroom, agreeing on some convenient set of axes (e.g., an intersection of walls and floor) and some units in the x, y, and z directions (e.g., using lengths of sides of floor, ceiling, or wall tiles). Let O be the origin of your coordinate system. Then, choose three points, A, B, and C in the room, and complete the following.

(a) Determine the coordinates of the points A, B, and C.

(b) Determine the components of the indicated vectors.

$$\text{(i) } \overrightarrow{OA} \quad \text{(ii) } \overrightarrow{OB} \quad \text{(iii) } \overrightarrow{OC} \quad \text{(iv) } \overrightarrow{AB} \quad \text{(v) } \overrightarrow{AC} \quad \text{(vi) } \overrightarrow{BC}$$

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Activity 9.9.

Let $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -1, 4\rangle$.

(a) Using the two specific vectors above, what is the natural way to define the vector sum $\mathbf{u} + \mathbf{v}$?

(b) In general, how do you think the vector sum $\mathbf{a}+\mathbf{b}$ of vectors $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ in $\mathbb{R}^2$ should be defined? Write a formal definition of a vector sum based on your intuition.

(c) In general, how do you think the vector sum $\mathbf{a} + \mathbf{b}$ of vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ in $\mathbb{R}^3$ should be defined? Write a formal definition of a vector sum based on your intuition.

(d) Returning to the specific vector $\mathbf{v} = \langle -1, 4\rangle$ given above, what is the natural way to define the scalar multiple $\frac{1}{2}\mathbf{v}$?

(e) In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2\rangle$ in $\mathbb{R}^2$ by a scalar c should be defined? how about for a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ in $\mathbb{R}^3$ by a scalar c? Write a formal definition of a scalar multiple of a vector based on your intuition.

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Activity 9.10.

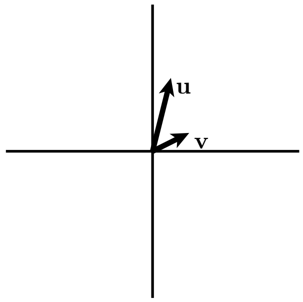

Figure 9.5

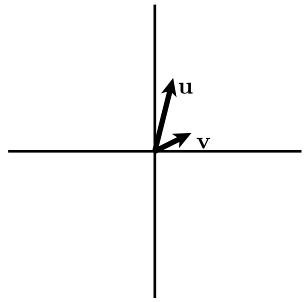

Figure 9.6

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in Figure 9.5.

(a) On Figure 9.5, sketch the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{v}-\mathbf{u}$, $2\mathbf{u}$, $-2\mathbf{u}$, and $-3\mathbf{v}$.

(b) What is $0\mathbf{v}$?

(c) On Figure 9.6, sketch the vectors $-3\mathbf{v}$, $-2\mathbf{v}$, $-1\mathbf{v}$, $2\mathbf{v}$, and $3\mathbf{v}$.

(d) Give a geometric description of the set of vectors $t\mathbf{v}$ where t is any scalar.

(e) On Figure 9.6, sketch the vectors $\mathbf{u}-3\mathbf{v}$, $\mathbf{u}-2\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{u}+2\mathbf{v}$.

(f) Give a geometric description of the set of vectors $\mathbf{u}+t\mathbf{v}$ where t is any scalar.

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Activity 9.11.

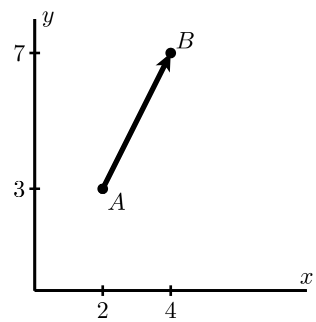

Figure 9.7: The vector defined by A and B.

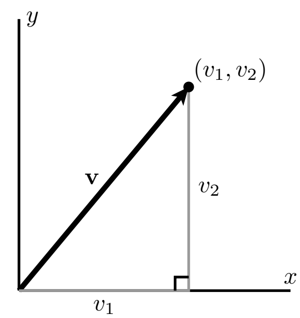

Figure 9.8: An arbitrary vector, $\mathbf{v}$.

(a) Let $A = (2, 3)$ and $B = (4, 7)$, as shown in Figure 9.7. Compute $|\overrightarrow{AB}|$.

(b) Let $\mathbf{v} = \langle v_1, v_2 \rangle$ be the vector in $\mathbb{R}^2$ with components v_1 and v_2 as shown in Figure 9.8. Use the distance formula to find a general formula for $|\mathbf{v}|$.

(c) Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be a vector in $\mathbb{R}^3$. Use the distance formula to find a general formula for $|\mathbf{v}|$.

(d) Suppose that $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$. Find $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{u} + \mathbf{v}|$. Is it true that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

(e) Under what conditions will $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$? (Hint: Think about how $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ form the sides of a triangle.)

(f) With the vector $\mathbf{u} = \langle 2, 3 \rangle$, find the lengths of $2\mathbf{u}$, $3\mathbf{u}$, and $-2\mathbf{u}$, respectively, and use proper notation to label your results.

(g) If t is any scalar, how is $|t\mathbf{u}|$ related to $|\mathbf{u}|$?

(h) A **unit vector** is a vector whose magnitude is 1. Of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{i} + \mathbf{j}$, which are unit vectors?

(i) Find a unit vector $\mathbf{v}$ whose direction is the same as $\mathbf{u} = \langle 2, 3 \rangle$. (Hint: Consider the result of part (g).)

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9.3 The Dot Product

Preview Activity 9.3. For two-dimensional vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, the dot product is simply the scalar obtained by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

(a) If $\mathbf{u} = \langle 3, 4 \rangle$ and $\mathbf{v} = \langle -2, 1 \rangle$, find the dot product $\mathbf{u} \cdot \mathbf{v}$.

(b) Find $\mathbf{i} \cdot \mathbf{i}$ and $\mathbf{i} \cdot \mathbf{j}$.

(c) If $\mathbf{u} = \langle 3, 4 \rangle$, find $\mathbf{u} \cdot \mathbf{u}$. How is this related to $|\mathbf{u}|$?

(d) On the axes in Figure 9.9, plot the vectors $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$. Then, find $\mathbf{u} \cdot \mathbf{v}$. What is the angle between these vectors?

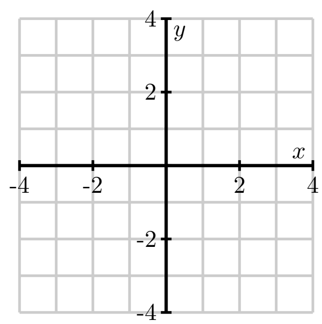

Figure 9.9: For part (d)

(e) On the axes in Figure 9.10, plot the vector $\mathbf{u} = \langle 1, 3 \rangle$.

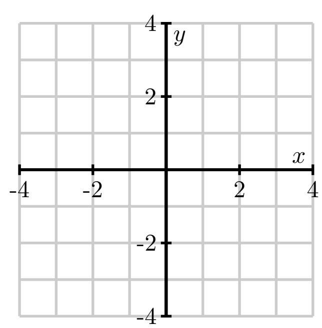

Figure 9.10: For part (e)

For each of the following vectors $\mathbf{v}$, plot the vector on Figure 9.10 and then compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

- $\mathbf{v} = \langle 3, 2 \rangle$.
- $\mathbf{v} = \langle 3, 0 \rangle$.
- $\mathbf{v} = \langle 3, -1 \rangle$.
- $\mathbf{v} = \langle 3, -2 \rangle$.
- $\mathbf{v} = \langle 3, -4 \rangle$.

(f) Based upon the previous part of this activity, what do you think is the sign of the dot product in the following three cases shown in Figure 9.11?

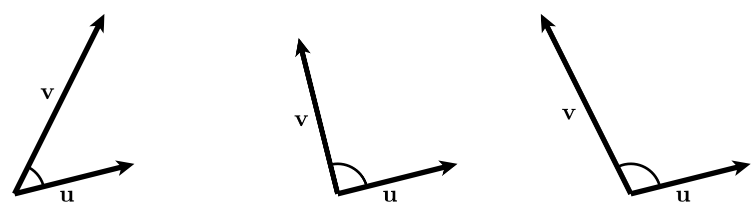

Figure 9.11: For part (f)

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Activity 9.12.

Determine each of the following.

(a) $\langle 1, 2, -3\rangle \cdot \langle 4, -2, 0\rangle$.

(b) $\langle 0, 3, -2, 1\rangle \cdot \langle 5, -6, 0, 4\rangle$

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Activity 9.13.

Determine each of the following.

(a) The length of the vector $\mathbf{u} = \langle 1, 2, -3 \rangle$ using the dot product.

(b) The angle between the vectors $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle 4, -1 \rangle$ to the nearest tenth of a degree.

(c) The angle between the vectors $\mathbf{y} = \langle 1, 2, -3 \rangle$ and $\mathbf{z} = \langle -2, 1, 1 \rangle$ to the nearest tenth of a degree.

(d) If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is a right angle, what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ say about their dot product?

(e) If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is acute—that is, less than $\pi/2$—what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ say about their dot product?

(f) If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is obtuse—that is, greater than $\pi/2$—what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ say about their dot product?

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Activity 9.14.

Determine the work done by a 25 pound force acting at a 30° angle to the direction of the object's motion, if the object is pulled 10 feet. In addition, is more work or less work done if the angle to the direction of the object's motion is 60°?

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Activity 9.15.

Let $\mathbf{u} = \langle 2, 6 \rangle$ and $\mathbf{v} = \langle 4, -8 \rangle$. Find $\text{comp}_{\mathbf{v}}\mathbf{u}$, $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\perp \mathbf{v}}\mathbf{u}$, and draw a picture to illustrate. Finally, express $\mathbf{u}$ as the sum of two vectors where one is parallel to $\mathbf{v}$ and the other is perpendicular to $\mathbf{v}$.

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9.4 The Cross Product

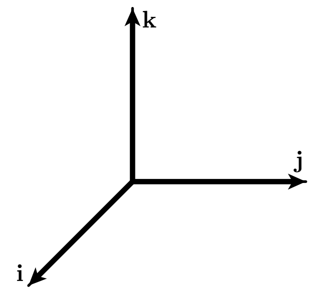

Figure 9.12: Basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Preview Activity 9.4. The cross product of two vectors, $\mathbf{u}$ and $\mathbf{v}$, will itself be a *vector* denoted $\mathbf{u} \times \mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule: if we point the index finger of our right hand in the direction of $\mathbf{u}$ and our middle finger in the direction of $\mathbf{v}$, then our thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.

(a) We begin by defining the cross products using the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Referring to Figure 9.12, explain why the definition $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ satisfies the right-hand rule.

(b) Now explain why the definition $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ satisfies the right-hand rule.

(c) Continuing in this way, complete the missing entries in Table 9.4.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$	$\mathbf{j} \times \mathbf{k} =$
$\mathbf{j} \times \mathbf{i} =$	$\mathbf{k} \times \mathbf{i} =$	$\mathbf{k} \times \mathbf{j} =$

Table 9.4: Table of cross products involving $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

(d) Up to this point, the products you have seen, such as the product of real numbers and the dot product of vectors, have been commutative, meaning that the product does not depend on the order of the terms. For instance, $2 \cdot 5 = 5 \cdot 2$. The table above suggests, however, that the cross product is *anti-commutative*: for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

If we consider the case when $\mathbf{u} = \mathbf{v}$, this shows that $\mathbf{v} \times \mathbf{v} = -\mathbf{v} \times \mathbf{v}$. What does this tell us about $\mathbf{v} \times \mathbf{v}$; in particular, what vector is unchanged by scalar multiplication by -1?

(e) The cross product is also a *bilinear* operation, meaning that it interacts with scalar multiplication and vector addition as one would expect: $(c\mathbf{u} + \mathbf{v}) \times \mathbf{w} = c(\mathbf{u} \times \mathbf{w}) + \mathbf{v} \times \mathbf{w}$. Using this property along with Table 9.4, find the cross product $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{k}$.

(f) Verify that the cross product $\mathbf{u} \times \mathbf{v}$ you just found in part (e) is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(g) Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ in the xy-plane as shown below in Figure 9.13.

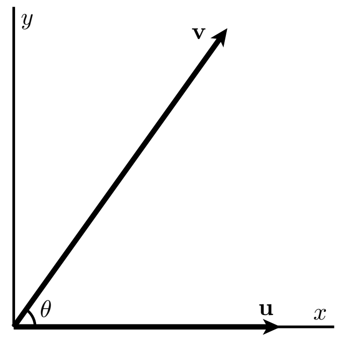

Figure 9.13: Two vectors in the xy-plane

Explain why $\mathbf{u} = |\mathbf{u}|\mathbf{i}$ and $\mathbf{v} = |\mathbf{v}| \cos\theta\mathbf{i} + |\mathbf{v}| \sin\theta\mathbf{j}$. Then compute the length of $|\mathbf{u} \times \mathbf{v}|$.

(h) Multiplication of real numbers is *associative*, which means, for instance, that $(2 \cdot 5) \cdot 3 = 2 \cdot (5 \cdot 3)$. Is it true that the cross product of vectors is associative? For instance, is it true that $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$?

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Activity 9.16.

Suppose $\mathbf{u} = \langle 2, -1, 0 \rangle$ and $\mathbf{v} = \langle 0, 1, 3 \rangle$. Use the formula (**??**) for the following.

(a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

(b) Find the cross product $\mathbf{u} \times \mathbf{i}$.

(c) Find the cross product $\mathbf{u} \times \mathbf{u}$.

(d) Evaluate the dot products $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$. What does this tell you about the geometric relationship among $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$?

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Activity 9.17.

(a) Find the area of the parallelogram formed by the vectors $\mathbf{u} = \langle 1, 3, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$.

(b) Find the area of the parallelogram in $\mathbb{R}^3$ whose vertices are $(1, 0, 1)$, $(0, 0, 1)$, $(2, 1, 0)$, and $(1, 1, 0)$.

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Activity 9.18.

Suppose $\mathbf{u} = \langle 3, 5, -1 \rangle$ and $\mathbf{v} = \langle 2, -2, 1 \rangle$.

(a) Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the volume of the parallelepiped formed by the vector $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} = \langle 3, 3, 1 \rangle$.

(c) Find a vector orthogonal to the plane containing the points $(0, 1, 2)$, $(4, 1, 0)$, and $(-2, 2, 2)$.

(d) Given the vectors $\mathbf{u}$ and $\mathbf{v}$ shown below in Figure 9.14, sketch the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

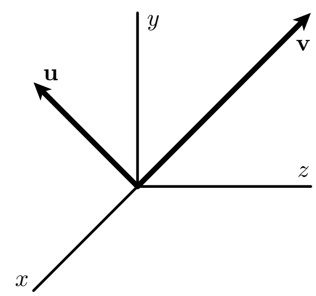

Figure 9.14: Vectors $\mathbf{u}$ and $\mathbf{v}$

(e) Do the vectors $\mathbf{u} = \langle 1, 3, -2 \rangle$, $\mathbf{v} = \langle 2, 1, -4 \rangle$, and $\mathbf{w} = \langle 0, 1, 0 \rangle$ lie in the same plane? Use the concepts from this section to explain.

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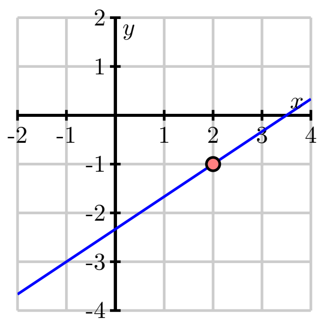
9.5 Lines and Planes in Space

Preview Activity 9.5. We are familiar with equations of lines in the plane in the form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept. In this activity, we explore a more flexible way of representing lines that we can use not only in the plane, but in higher dimensions as well.

To begin, consider the line through the point $(2, -1)$ with slope $\frac{2}{3}$.



(a) Suppose we increase x by 1 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate 3?

(b) Suppose we decrease x by 3.25 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate -1.25?

(c) Now, suppose we increase x by some arbitrary value $3t$ from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate $2 + 3t$?

(d) Observe that the slope of the line is related to any vector whose y-component divided by the x-component is the slope of the line. For the line in this exercise, we might use the vector $\langle 3, 2\rangle$, which describes the direction of the line. Explain why the terminal points of the vectors $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = \langle 2, -1\rangle + \langle 3, 2\rangle t,$$

trace out the graph of the line through the point $(2, -1)$ with slope $\frac{2}{3}$.

(e) Now we extend this vector approach to $\mathbb{R}^3$ and consider a second example. Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through the point $(1, 0, 2)$ in the direction of the vector $\langle 2, -1, 4\rangle$.

Find the coordinates of three distinct points on line $\mathcal{L}$. Explain your thinking.

(f) Find a vector in form

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + \langle a, b, c\rangle t$$

whose terminal points trace out the line $\mathcal{L}$ that is described in (e). That is, you should be able to locate any point on the line by determining a corresponding value of t.

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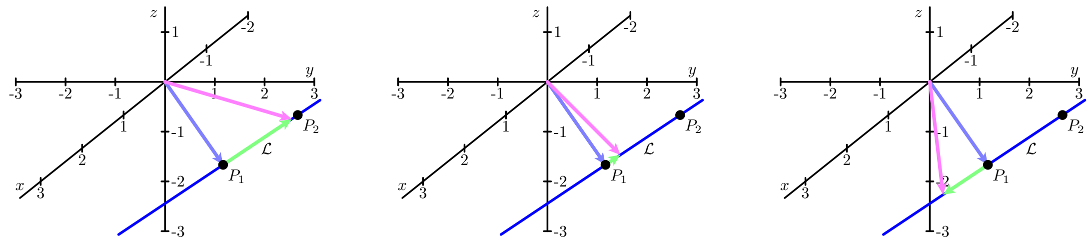

Figure 9.15: A line in 3-space.

Activity 9.19.

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$. Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2, and note that three snapshots of this line are shown in Figure 9.15.

(a) Find a direction vector for the line $\mathcal{L}$.

(b) Find a vector equation of $\mathcal{L}$ in the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

(c) Consider the vector equation $\mathbf{s}(t) = \langle -5, 0, -3 \rangle + t\langle 6, 2, 2 \rangle$. What is the direction of the line given by $\mathbf{s}(t)$? Is this new line parallel to line $\mathcal{L}$?

(d) Do $\mathbf{r}(t)$ and $\mathbf{s}(t)$ represent the same line, $\mathcal{L}$? Explain.

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Activity 9.20.

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2, which is the same line as in Activity 9.19.

(a) Find the parametric equations of the line $\mathcal{L}$.

(b) Does the point $(1, 2, 1)$ lie on $\mathcal{L}$? If so, what value of t results in this point?

(c) Consider another line, $\mathcal{K}$, whose parametric equations are

$$x(s) = -2 + 4s, \;\; y(s) = 1 - 3s, \;\; -2 + 2s.$$

What is the direction of line $\mathcal{K}$?

(d) Do lines $\mathcal{L}$ and $\mathcal{K}$ intersect? If so, provide the point of intersection and the t and s values, respectively, that result in the point. If not, explain why.

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Activity 9.21.

(a) Write the equation of the plane p_1 passing through the point $(0, 2, 4)$ and perpendicular to the vector $\mathbf{n} = \langle 2, -1, 1 \rangle$.

(b) Is the point $(2, 0, 2)$ on the plane p_1?

(c) Write the equation of the plane p_2 that is parallel to p_1 and passing through the point $(3, 0, 4)$.

(d) Write the parametric description of the line l passing through the point $(2, 0, 2)$ and perpendicular to the plane p_3 described the equation $x + 2y - 2z = 7$.

(e) Find the point at which l intersects the plane p_3.

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Activity 9.22.

Let $P_0 = (1, 2, -1)$, $P_1 = (1, 0, -1)$, and $P_2 = (0, 1, 3)$ and let p be the plane containing P_0, P_1, and P_2.

(a) Determine the components of the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$.

(b) Find a normal vector $\mathbf{n}$ to the plane p.

(c) Find the scalar equation of the plane p.

(d) Consider a second plane, q, whose scalar equation is $-3(x-1)+4(y+3)+2(z-5) = 0$. Find two different points on plane q, as well as a vector $\mathbf{m}$ that is normal to q.

(e) We define the angle between two planes to be the angle between their respective normal vectors. What is the angle between planes p and q?

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9.6 Vector-Valued Functions

Preview Activity 9.6. In this activity we consider how we might use vectors to define a curve in space.

(a) On a single set of axes in $\mathbb{R}^2$, draw the vectors $\langle \cos(0), \sin(0) \rangle$, $\left\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right) \right\rangle$, $\langle \cos(\pi), \sin(\pi) \rangle$, and $\left\langle \cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right) \right\rangle$ with their initial points at the origin.

(b) On the same set of axes, draw the vectors $\left\langle \cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right) \right\rangle$, $\left\langle \cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right) \right\rangle$, $\left\langle \cos\left(\frac{5\pi}{4}\right), \sin\left(\frac{5\pi}{4}\right) \right\rangle$, and $\left\langle \cos\left(\frac{7\pi}{4}\right), \sin\left(\frac{7\pi}{4}\right) \right\rangle$ with their initial points at the origin.

(c) Based on the pictures from parts (a) and (b), sketch the set of *terminal* points of all of the vectors of the form $\langle \cos(t), \sin(t) \rangle$, where t assumes values from 0 to 2π. What is the resulting figure? Why?

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Activity 9.23.

The same curve can be represented with different parameterizations. Use your calculator,[2] Wolfram|Alpha, or some other graphing device[3] to plot the curves generated by the following vector-valued functions. Compare and contrast the graphs – explain how they are alike and how they are different.

(a) $\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$

(b) $\mathbf{r}(t) = \langle \sin(2t), \cos(2t) \rangle$

(c) $\mathbf{r}(t) = \langle \cos(t+\pi), \sin(t+\pi) \rangle$

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[2] If you have a graphing calculator you can draw graphs of vector-valued functions in $\mathbb{R}^2$ using the parametric mode (often found in the MODE menu).

[3] e.g., `http://webspace.ship.edu/msrenault/ggb/parametric_grapher.html`

Activity 9.24.

Vector-valued functions can be used to generate many interesting curves. Graph each of the following using an appropriate tool[4], and then write one sentence for each function to describe the behavior of the resulting curve.

(a) $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$

(b) $\mathbf{r}(t) = \langle \sin(t)\cos(t), t\sin(t)\rangle$

(c) $\mathbf{r}(t) = \langle t^2\sin(t)\cos(t), 0.9t\cos(t^2), \sin(t)\rangle$

(d) $\mathbf{r}(t) = \langle \sin(5t), \sin(4t)\rangle$

(e) Experiment with different formulas for $x(t)$ and $y(t)$ and ranges for t to see what other interesting curves you can generate. Share your best results with peers.

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[4]e.g., the 2D grapher at http://webspace.ship.edu/msrenault/ggb/parametric_grapher.html, or for 3D graphs Wolfram|Alpha, an on-line 3D grapher like http://www.math.uri.edu/~bkaskosz/flashmo/parcur/, or some other device

Activity 9.25.

Consider the paraboloid defined by $f(x, y) = x^2 + y^2$.

(a) Find a parameterization for the $x = 2$ trace of f. What type of curve does this trace describe?

(b) Find a parameterization for the $y = -1$ trace of f. What type of curve does this trace describe?

(c) Find a parameterization for the level curve $f(x, y) = 25$. What type of curve does this trace describe?

(d) How do your responses change to all three of the preceding question if you instead consider the function g defined by $g(x, y) = x^2 - y^2$? (Hint for generating one of the parameterizations: $\sec^2(t) - \tan^2(t) = 1$.)

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9.7 Derivatives and Integrals of Vector-Valued Functions

Preview Activity 9.7. Let $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(2t)\mathbf{j}$ describe the path traveled by an object at time t.

(a) Use appropriate technology to help you sketch the graph of the vector-valued function $\mathbf{r}(t)$, and then locate and label the point on the graph when $t = \pi$.

(b) Recall that for functions of a single variable, the derivative of a sum is the sum of the derivatives; that is, $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$. With this idea in mind and viewing **i** and **j** as constant vectors, what do you expect the derivative of **r** to be? Write a proposed formula for $\mathbf{r}'(t)$.

(c) Use your result from part (b) to compute $\mathbf{r}'(\pi)$. Sketch this vector $\mathbf{r}'(\pi)$ as emanating from the point on the graph when $t = \pi$, and explain what you think $\mathbf{r}'(\pi)$ tells us about the object's motion.

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Activity 9.26.

Let's investigate how we can interpret the derivative $\mathbf{r}'(t)$. Let $\mathbf{r}$ be the vector-valued function whose graph is shown in Figure 9.16, and let h be a scalar that represents a small change in time. The vector $\mathbf{r}(t)$ is the blue vector in Figure 9.16 and $\mathbf{r}(t+h)$ is the green vector.

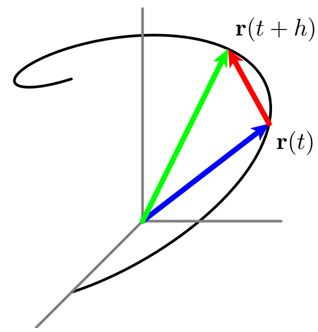

Figure 9.16: A single difference quotient.

(a) Is the quantity $\mathbf{r}(t+h) - \mathbf{r}(t)$ a vector or a scalar? Identify this object in Figure 9.16.

(b) Is $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ a vector or a scalar? Sketch a representative vector $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ with $h < 1$ in Figure 9.16.

(c) Think of $\mathbf{r}(t)$ as providing the position of an object moving along the curve these vectors trace out. What do you think that the vector $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ measures? Why? (Hint: You might think analogously about difference quotients such as $\frac{f(x+h)-f(x)}{h}$ or $\frac{s(t+h)-s(t)}{h}$ from calculus I.)

(d) Figure 9.17 presents three snapshots of the vectors $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ as we let $h \to 0$. Write 2-3 sentences to describe key attributes of the vector

$$\lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

(Hint: Compare to limits such as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ or $\lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$ from calculus I, keeping in mind that in three dimensions there is no general concept of slope.)

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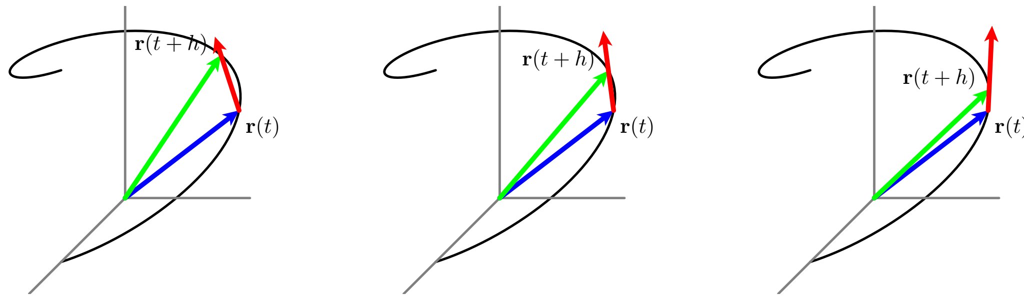

Figure 9.17: Snapshots of several difference quotients.

Activity 9.27.

For each of the following vector-valued functions, find $\mathbf{r}'(t)$.

(a) $\mathbf{r}(t) = \langle \cos(t), t\sin(t), \ln(t)\rangle$.

(b) $\mathbf{r}(t) = \langle t^2 + 3t, e^{-2t}, \frac{t}{t^2+1}\rangle$.

(c) $\mathbf{r}(t) = \langle \tan(t), \cos(t^2), te^{-t}\rangle$.

(d) $\mathbf{r}(t) = \langle \sqrt{t^4+4}, \sin(3t), \cos(4t)\rangle$.

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Activity 9.28.

The left side of figure 9.18 shows the curve described by the vector-valued function

$$\mathbf{r}(t) = \left\langle 2t - \frac{1}{2}t^2 + 1, t - 1 \right\rangle.$$

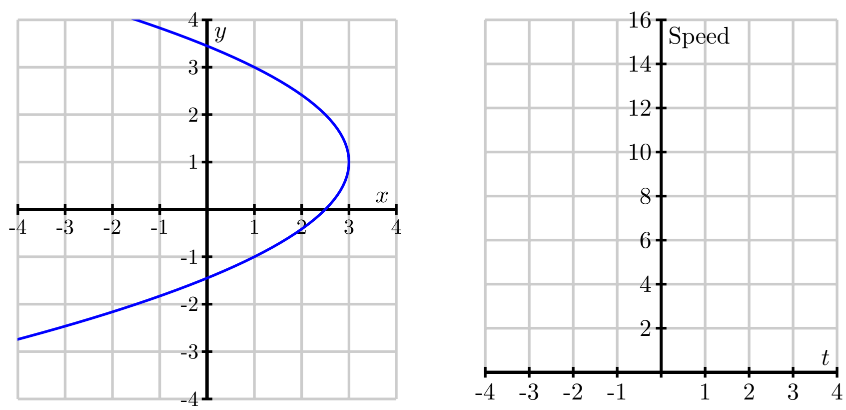

Figure 9.18: The curve $\mathbf{r}(t) = \left\langle 2t - \frac{1}{2}t^2 + 1, t - 1 \right\rangle$ and its speed.

(a) Find the object's velocity $\mathbf{v}(t)$.

(b) Find the object's acceleration $\mathbf{a}(t)$.

(c) Indicate on the left of Figure 9.18 the object's position, velocity and acceleration at the times $t = 0, 2, 4$. Draw the velocity and acceleration vectors with their tails placed at the object's position.

(d) Recall that the speed is $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Find the object's speed and graph it as a function of time t on the right of Figure 9.18. When is the object's speed the slowest? When is the speed increasing? When it is decreasing?

(e) What seems to be true about the angle between $\mathbf{v}$ and $\mathbf{a}$ when the speed is at a minimum? What is the angle between $\mathbf{v}$ and $\mathbf{a}$ when the speed is increasing? when the speed is decreasing?

(f) Since the square root is an increasing function, we see that the speed increases precisely when $\mathbf{v} \cdot \mathbf{v}$ is increasing. Use the product rule for the dot product to express $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$ in terms of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Use this to explain why the speed is increasing when $\mathbf{v} \cdot \mathbf{a} > 0$ and decreasing when $\mathbf{v} \cdot \mathbf{a} < 0$. Compare this to part (d).

(g) Show that the speed's rate of change is

$$\frac{d}{dt}|\mathbf{v}(t)| = \text{comp}_{\mathbf{v}}\mathbf{a}.$$

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Activity 9.29.

Let
$$\mathbf{r}(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + t\mathbf{k}.^{5}$$

(a) Determine the coordinates of the point on the curve traced out by $\mathbf{r}(t)$ when $t = \pi$.

(b) Find a direction vector for the line tangent to the graph of $\mathbf{r}$ at the point where $t = \pi$.

(c) Find the parametric equations of the line tangent to the graph of $\mathbf{r}$ when $t = \pi$.

(d) Sketch a plot of the curve $\mathbf{r}(t)$ and its tangent line near the point where $t = \pi$. In addition, include a sketch of $\mathbf{r}'(\pi)$. What is the important role of $\mathbf{r}'(\pi)$ in this activity?

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[5] You can sketch the graph with Wolfram Alpha, the applet at http://gvsu.edu/s/LR, or some other appropriate technology.

Activity 9.30.

Suppose a moving object in space has its velocity given by

$$\mathbf{v}(t) = (-2\sin(2t))\mathbf{i} + (2\cos(t))\mathbf{j} + \left(1 - \frac{1}{1+t}\right)\mathbf{k}.$$

A graph of the position of the object for times t in $[-0.5, 3]$ is shown in Figure 9.19. Suppose further that the object is at the point $(1.5, -1, 0)$ at time $t = 0$.

(a) Determine $\mathbf{a}(t)$, the acceleration of the object at time t.

(b) Determine $\mathbf{r}(t)$, position of the object at time t.

(c) Compute and sketch the position, velocity, and acceleration vectors of the object at time $t = 1$, using Figure 9.19.

(d) Finally, determine the vector equation for the tangent line, $\mathbf{L}(t)$, that is tangent to the position curve at $t = 1$.

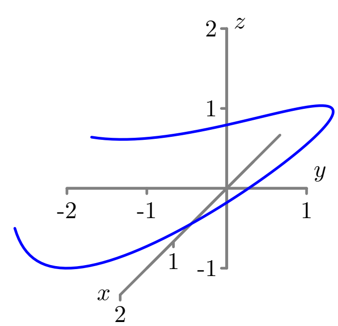

Figure 9.19: The position graph for the function in Activity 9.30.

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9.8 Arc Length and Curvature

Preview Activity 9.8. In earlier investigations, we have used integration to calculate quantities such as area, volume, mass, and work. We are now interested in determining the length of a space curve.

Consider the smooth curve in 3-space defined by the vector-valued function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle \cos(t), \sin(t), t \rangle$$

for t in the interval $[0, 2\pi]$. Pictures of the graph of $\mathbf{r}$ are shown in Figure 9.20. We will use the integration process to calculate the length of this curve. In this situation we partition the interval $[0, 2\pi]$ into n subintervals of equal length and let $0 = t_0 < t_1 < t_2 < \cdots < t_n = b$ be the endpoints of the subintervals. We then approximate the length of the curve on each subinterval with some related quantity that we can compute. In this case, we approximate the length of the curve on each subinterval with the length of the segment connecting the endpoints. Figure 9.20 illustrates the process in three different instances using increasing values of n.

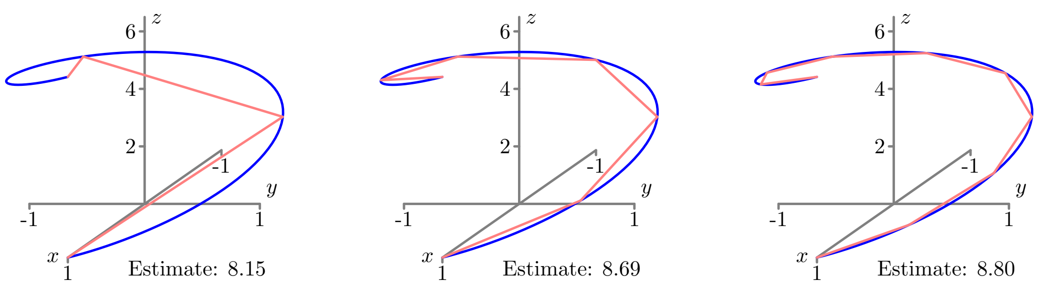

Figure 9.20: Approximating the length of the curve with $n = 3$, $n = 6$, and $n = 9$.

(a) Write a formula for the length of the line segment that connects the endpoints of the curve on the ith subinterval $[t_{i-1}, t_i]$. (This length is our approximation of the length of the curve on this interval.)

(b) Use your formula in part (a) to write a sum that adds all of the approximations to the lengths on each subinterval.

(c) What do we need to do with the sum in part (b) in order to obtain the exact value of the length of the graph of $\mathbf{r}(t)$ on the interval $[0, 2\pi]$?

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Activity 9.31.

Here we calculate the arc length of two familiar curves.

(a) Use Equation (**??**) to calculate the circumference of a circle of radius r.

(b) Find the exact length of the spiral defined by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ on the interval $[0, 2\pi]$.

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Activity 9.32.

Let $y = f(x)$ define a smooth curve in 2-space. Parameterize this curve and use Equation (**??**) to show that the length of the curve define by f on an interval $[a, b]$ is

$$\int_a^b \sqrt{1 + [f'(t)]^2}\, dt.$$

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Activity 9.33.

In this activity we parameterize a line in 2-space in terms of arc length. Consider the line with parametric equations

$$x(t) = x_0 + at \quad \text{and} \quad y(t) = y_0 + bt.$$

(a) To write t in terms of s, evaluate the integral

$$s = L(t) = \int_0^t \sqrt{(x'(w))^2 + (y'(w))^2}\, dw$$

to determine the length of the line from time 0 to time t.

(b) Use the formula from (a) for s in terms of t to write t in terms of s. Then explain why a parameterization of the line in terms of arc length is

$$x(s) = x_0 + \frac{a}{\sqrt{a^2+b^2}}s \quad \text{and} \quad y(s) = y_0 + \frac{b}{\sqrt{a^2+b^2}}s. \tag{9.2}$$

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Activity 9.34.

Recall that an arc length parameterization of a circle in 2-space of radius a centered at the origin is, from (**??**),

$$\mathbf{r}(s) = \left\langle a\cos\left(\frac{s}{a}\right),\ a\sin\left(\frac{s}{a}\right)\right\rangle.$$

Show that the curvature of this circle is the constant $\frac{1}{a}$. What can you say about the relationship between the size of the radius of a circle and the value of its curvature? Why does this make sense?

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Activity 9.35.

Use one of the two formulas for κ in terms of t to help you answer the following questions.

(a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has parameterization

$$\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle.$$

Find the curvature of the ellipse. Assuming $0 < b < a$, at what points is the curvature the greatest and at what points is the curvature the smallest? Does this agree with your intuition?

(b) The standard helix has parameterization $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$. Find the curvature of the helix. Does the result agree with your intuition?

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Chapter 10

Derivatives of Multivariable Functions

10.1 Limits

Preview Activity 10.1. We investigate the limits of several different functions by working with tables and graphs.

(a) Consider the function f defined by

$$f(x) = 3 - x.$$

Complete the following table of values.

x	$f(x)$
-0.2	
-0.1	
0.0	
0.1	
0.2	

What does the table suggest regarding $\lim_{x\to 0} f(x)$?

(b) Explain how your results in (a) are reflected in Figure 10.1.

(c) Next, consider

$$g(x) = \frac{x}{|x|}.$$

Complete the following table of values near $x = 0$, the point at which g is not defined.

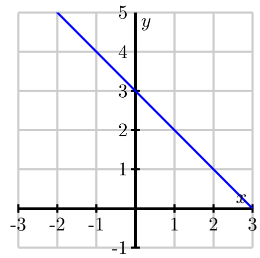

Figure 10.1: The graph of $f(x) = 3 - x$.

x	$g(x)$
-0.1	
-0.01	
-0.001	
0.001	
0.01	
0.1	

What does this suggest about $\lim_{x\to 0} g(x)$?

(d) Explain how your results in (c) are reflected in Figure 10.2.

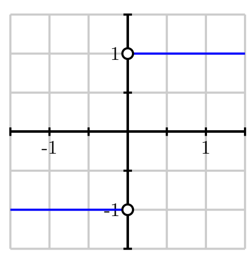

Figure 10.2: The graph of $g(x) = \frac{x}{|x|}$.

(e) Now, let's examine a function of two variables. Let

$$f(x, y) = 3 - x - 2y$$

and complete the following table of values.

$x \backslash y$	-1	-0.1	0	0.1	1
-1					
-0.1					
0					
0.1					
1					

What does the table suggest about $\lim_{(x,y)\to(0,0)} f(x, y)$?

(f) Explain how your results in (e) are reflected in Figure 10.3. Compare this limit to the limit in part (a). How are the limits similar and how are they different?

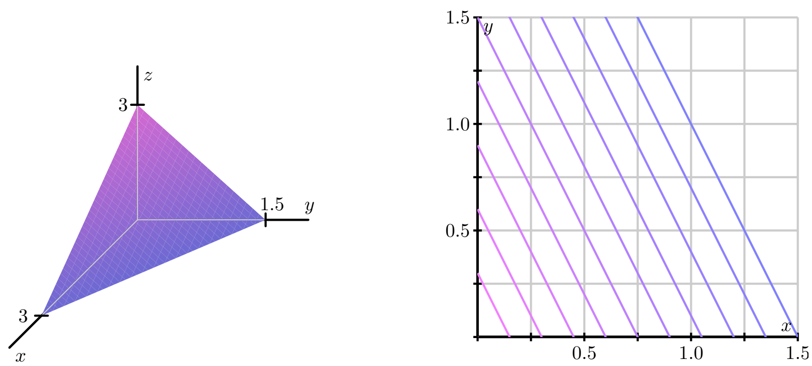

Figure 10.3: At left, the graph of $f(x, y) = 3 - x - 2y$; at right, its contour plot.

(g) Finally, consider

$$g(x, y) = \frac{2xy}{x^2 + y^2},$$

which is not defined at $(0, 0)$, and complete the following table of values of $g(x, y)$.

$x \backslash y$	-1	-0.1	0	0.1	1
-1					
-0.1					
0			—		
0.1					
1					

What does this suggest about the $\lim_{(x,y)\to(0,0)} g(x, y)$?

(h) Explain how your results are reflected in Figure 10.4. Compare this limit to the limit in part (b). How are the results similar and how are they different?

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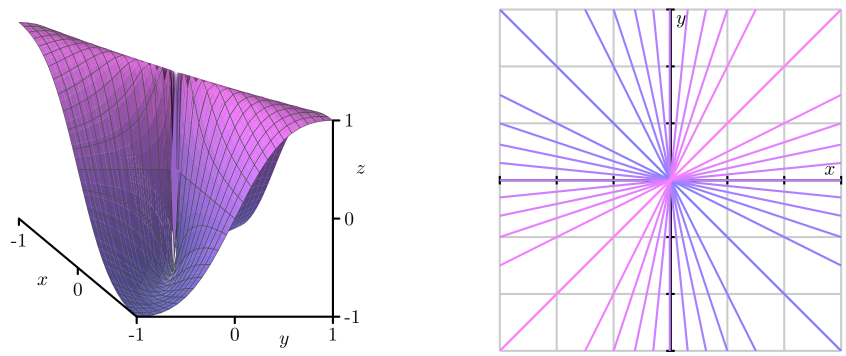

Figure 10.4: At left, the graph of $g(x,y) = \frac{2xy}{x^2+y^2}$; at right, its contour plot.

Activity 10.1.

Consider the function f, defined by

$$f(x, y) = \frac{y}{\sqrt{x^2 + y^2}},$$

whose graph is shown below in Figure 10.5

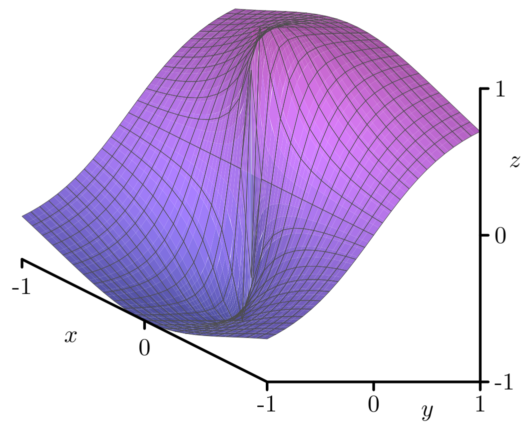

Figure 10.5: The graph of $f(x, y) = \frac{y}{\sqrt{x^2+y^2}}$.

(a) Is f defined at the point $(0, 0)$? What, if anything, does this say about whether f has a limit at the point $(0, 0)$?

(b) Values of f (to three decimal places) at several points close to $(0, 0)$ are shown in the table below.

$x \backslash y$	-1	-0.1	0	0.1	1
-1	-0.707	—	0	—	0.707
-0.1	—	-0.707	0	0.707	—
0	-1	-1	—	1	1
0.1	—	-0.707	0	0.707	—
1	-0.707	—	0	—	0.707

Based on these calculations, state whether f has a limit at $(0, 0)$ and give an argument supporting your statement. (Hint: The blank spaces in the table are there to help you see the patterns.)

(c) Now let's consider what happens if we restrict our attention to the x-axis; that is, consider what happens when $y = 0$. What is the behavior of $f(x, 0)$ as $x \to 0$? If we approach $(0, 0)$ by moving along the x-axis, what value do we find as the limit?

(d) What is the behavior of f along the line $y = x$ when $x > 0$; that is, what is the value of $f(x, x)$ when $x > 0$? If we approach $(0, 0)$ by moving along the line $y = x$ in the first quadrant (thus considering $f(x, x)$ as $x \to 0$, what value do we find as the limit?

(e) In general, if $\lim_{(x,y)\to(0,0)} f(x,y) = L$, then $f(x,y)$ approaches L as (x,y) approaches $(0,0)$, regardless of the path we take in letting $(x,y) \to (0,0)$. Explain what the last two parts of this activity imply about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$.

(f) Shown below in Figure 10.6 is a set of contour lines of the function f. What is the behavior of $f(x,y)$ as (x,y) approaches $(0,0)$ along any straight line? How does this observation reinforce your conclusion about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$ from the previous part of this activity?(Hint: Use the fact that a non-vertical line has equation $y = mx$ for some constant m.)

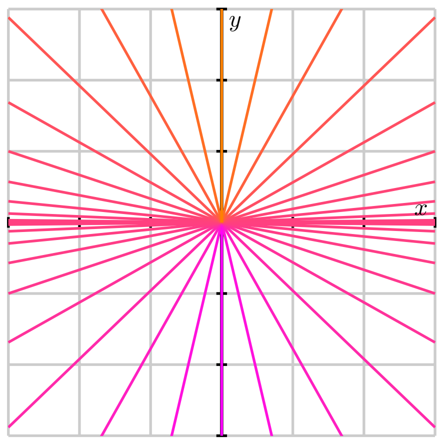

Figure 10.6: Contour lines of $f(x,y) = \frac{y}{\sqrt{x^2+y^2}}$.

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Activity 10.2.

Let's consider the function g defined by

$$g(x, y) = \frac{x^2 y}{x^4 + y^2}$$

and investigate the limit $\lim_{(x,y)\to(0,0)} g(x, y)$.

(a) What is the behavior of g on the x-axis? That is, what is $g(x, 0)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the x-axis?

(b) What is the behavior of g on the y-axis? That is, what is $g(0, y)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the y-axis?

(c) What is the behavior of g on the line $y = mx$? That is, what is $g(x, mx)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the line $y = mx$?

(d) Based on what you have seen so far, do you think $\lim_{(x,y)\to(0,0)} g(x, y)$ exists? If so, what do you think its value is?

(e) Now consider the behavior of g on the parabola $y = x^2$? What is $g(x, x^2)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along this parabola?

(f) State whether the limit $\lim_{(x,y)\to(0,0)} g(x, y)$ exists or not and provide a justification of your statement.

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10.2 First-Order Partial Derivatives

Preview Activity 10.2. Let's return to the function we considered in Preview Activity 9.1. Suppose we take out a \$18,000 car loan at interest rate r and we agree to pay off the loan in t years. The monthly payment, in dollars, is

$$M(r,t) = \frac{1500r}{1 - \left(1 + \frac{r}{12}\right)^{-12t}}.$$

(a) What is the monthly payment if the interest rate is $r = 3\% = 0.03$, and we pay the loan off in $t = 4$ years?

(b) Suppose the interest rate is fixed at $r = 3\% = 0.03$. Express M as a function f of t alone using $r = 0.03$. That is, let $f(t) = M(0.03, t)$. Sketch the graph of f on the left of Figure 10.7. Explain the meaning of the function f.

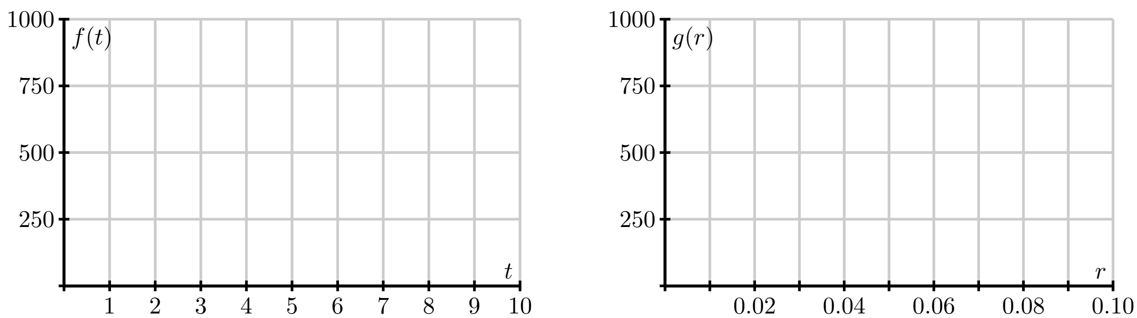

Figure 10.7: The graphs of $f(t) = M(0.03, t)$ and $g(r) = M(r, 4)$.

(c) Find the instantaneous rate of change $f'(4)$ and state the units on this quantity. What information does $f'(4)$ tell us about our car loan? What information does $f'(4)$ tell us about the graph you sketched in (b)?

(d) Express M as a function of r alone, using a fixed time of $t = 4$. That is, let $g(r) = M(r, 4)$. Sketch the graph of g on the right of Figure 10.7. Explain the meaning of the function g.

(e) Find the instantaneous rate of change $g'(0.03)$ and state the units on this quantity. What information does $g'(0.03)$ tell us about our car loan? What information does $g'(0.03)$ tell us about the graph you sketched in (d)?

⋈

Activity 10.3.

Consider the function f defined by

$$f(x,y) = \frac{xy^2}{x+1}$$

at the point $(1,2)$.

(a) Write the trace $f(x,2)$ at the fixed value $y = 2$. On the left side of Figure 10.8, draw the graph of the trace with $y = 2$ indicating the scale and labels on the axes. Also, sketch the tangent line at the point $x = 1$.

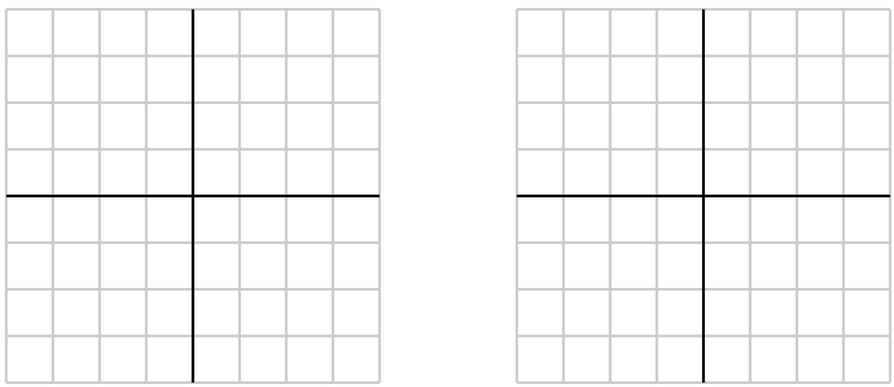

Figure 10.8: Traces of $f(x,y) = \frac{xy^2}{x+1}$.

(b) Find the partial derivative $f_x(1,2)$ and relate its value to the sketch you just made.

(c) Write the trace $f(1,y)$ at the fixed value $x = 1$. On the right side of Figure 10.8, draw the graph of the trace with $x = 1$ indicating the scale and labels on the axes. Also, sketch the tangent line at the point $y = 2$.

(d) Find the partial derivative $f_y(1,2)$ and relate its value to the sketch you just made.

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Activity 10.4.

(a) If we have the function f of the variables x and y and we want to find the partial derivative f_x, which variable do we treat as a constant? When we find the partial derivative f_y, which variable do we treat as a constant?

(b) If $f(x,y) = 3x^3 - 2x^2y^5$, find the partial derivatives f_x and f_y.

(c) If $f(x,y) = \dfrac{xy^2}{x+1}$, find the partial derivatives f_x and f_y.

(d) If $g(r,s) = rs\cos(r)$, find the partial derivatives g_r and g_s.

(e) Assuming $f(w,x,y) = (6w+1)\cos(3x^2 + 4xy^3 + y)$, find the partial derivatives f_w, f_x, and f_y.

(f) Find all possible first-order partial derivatives of $q(x,t,z) = \dfrac{x2^tz^3}{1+x^2}$.

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Activity 10.5.

The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D.$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

(a) State the units in which each of the partial derivatives, C_T, C_S and C_D, are expressed and explain the physical meaning of each.

(b) Find the partial derivatives C_T, C_S and C_D.

(c) Evaluate each of the three partial derivatives at the point where $T = 10$, $S = 35$ and $D = 100$. What does the sign of each partial derivatives tell us about the behavior of the function C at the point $(10, 35, 100)$?

$\triangleleft$

Activity 10.6.

The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In Table 10.2, the wind chill w, measured in degrees Fahrenheit, is a function of the wind speed v, measured in miles per hour, and the ambient air temperature T, also measured in degrees Fahrenheit. We thus view w as being of the form $w = w(v, T)$.

$v\backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

Table 10.1: Wind chill as a function of wind speed and temperature.

(a) Estimate the partial derivative $w_v(20, -10)$. What are the units on this quantity and what does it mean?

(b) Estimate the partial derivative $w_T(20, -10)$. What are the units on this quantity and what does it mean?

(c) Use your results to estimate the wind chill $w(18, -10)$.

(d) Use your results to estimate the wind chill $w(20, -12)$.

(e) Use your results to estimate the wind chill $w(18, -12)$.

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Activity 10.7.

Shown below in Figure 10.9 is a contour plot of a function f. The value of the function along a few of the contours is indicated to the left of the figure.

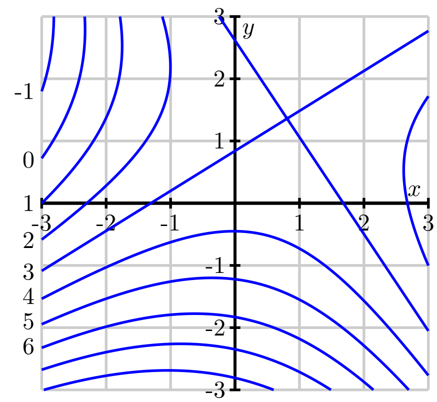

Figure 10.9: A contour plot of f.

(a) Estimate the partial derivative $f_x(-2, -1)$.

(b) Estimate the partial derivative $f_y(-2, -1)$.

(c) Estimate the partial derivatives $f_x(-1, 2)$ and $f_y(-1, 2)$.

(d) Locate one point (x, y) where the partial derivative $f_x(x, y) = 0$.

(e) Locate one point (x, y) where $f_x(x, y) < 0$.

(f) Locate one point (x, y) where $f_y(x, y) > 0$.

(g) Suppose you have a different function g, and you know that $g(2, 2) = 4$, $g_x(2, 2) > 0$, and $g_y(2, 2) > 0$. Using this information, sketch a possibility for the contour $g(x, y) = 4$ passing through $(2, 2)$ on the left side of Figure 10.10. Then include possible contours $g(x, y) = 3$ and $g(x, y) = 5$.

(h) Suppose you have yet another function h, and you know that $h(2, 2) = 4$, $h_x(2, 2) < 0$, and $h_y(2, 2) > 0$. Using this information, sketch a possible contour $h(x, y) = 4$ passing through $(2, 2)$ on the right side of Figure 10.10. Then include possible contours $h(x, y) = 3$ and $h(x, y) = 5$.

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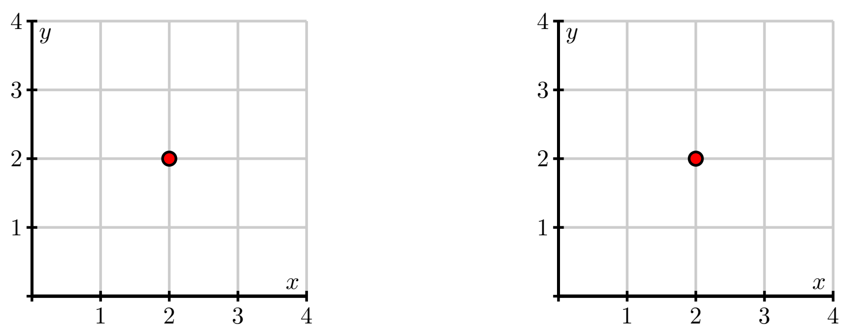

Figure 10.10: Plots for contours of g and h.

10.3 Second-Order Partial Derivatives

Preview Activity 10.3. Once again, let's consider the function f defined by $f(x, y) = \frac{x^2 \sin(2y)}{32}$ that

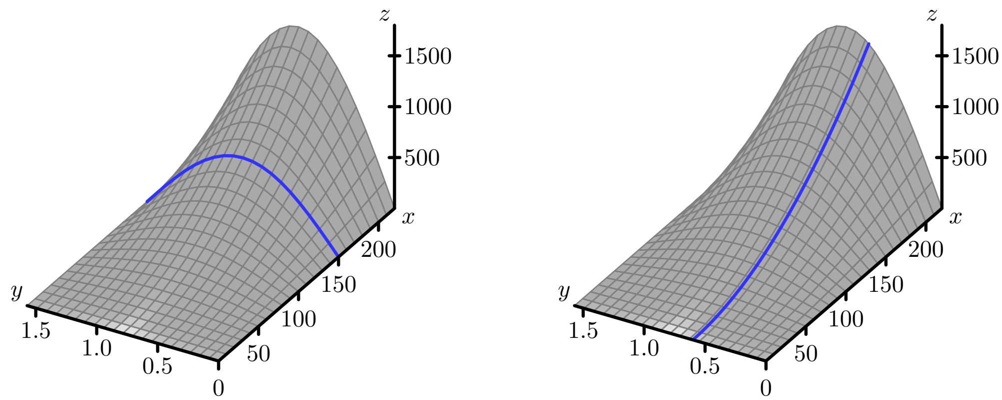

Figure 10.11: The range function with traces $y = 0.6$ and $x = 150$.

measures a projectile's range as a function of its initial speed x and launch angle y. The graph of this function, including traces with $x = 150$ and $y = 0.6$, is shown in Figure 10.11.

(a) Compute the partial derivative f_x and notice that f_x itself is a new function of x and y.

(b) We may now compute the partial derivatives of f_x. Find the partial derivative $f_{xx} = (f_x)_x$ and evaluate $f_{xx}(150, 0.6)$.

(c) Figure 10.12 shows the trace of f with $y = 0.6$ with three tangent lines included. Explain how your result from part (b) of this preview activity is reflected in this figure.

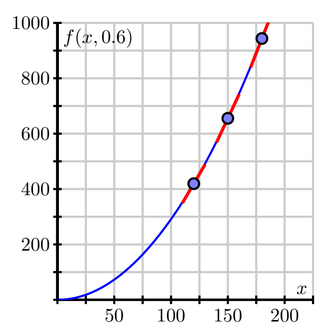

Figure 10.12: The trace with $y = 0.6$.

(d) Determine the partial derivative f_y, and then find the partial derivative $f_{yy} = (f_y)_y$. Evaluate $f_{yy}(150, 0.6)$.

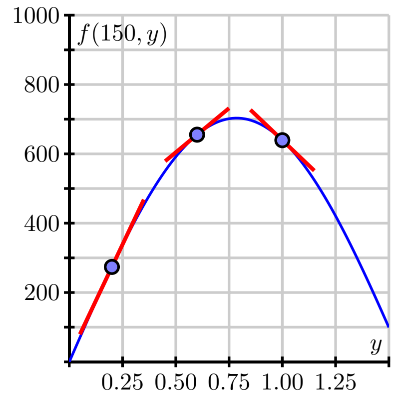

Figure 10.13: More traces of the range function.

(e) Figure 10.13 shows the trace $f(150, y)$ and includes three tangent lines. Explain how the value of $f_{yy}(150, 0.6)$ is reflected in this figure.

(f) Because f_x and f_y are each functions of both x and y, they each have two partial derivatives. Not only can we compute $f_{xx} = (f_x)_x$, but also $f_{xy} = (f_x)_y$; likewise, in addition to $f_{yy} = (f_y)_y$, but also $f_{yx} = (f_y)_x$. For the range function $f(x, y) = \frac{x^2 \sin(2y)}{32}$, use your earlier computations of f_x and f_y to now determine f_{xy} and f_{yx}. Write one sentence to explain how you calculated these "mixed" partial derivatives.

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Activity 10.8.

Find all second order partial derivatives of the following functions. For each partial derivative you calculate, state explicitly which variable is being held constant.

(a) $f(x, y) = x^2 y^3$

(b) $f(x, y) = y\cos(x)$

(c) $g(s, t) = st^3 + s^4$

(d) How many second order partial derivatives does the function h defined by $h(x, y, z) = 9x^9 z - xyz^9 + 9$ have? Find h_{xz} and h_{zx}.

$\triangleleft$

Activity 10.9.

We continue to consider the function f defined by $f(x, y) = \sin(x)e^{-y}$.

(a) In Figure 10.14, we see the trace of $f(x, y) = \sin(x)e^{-y}$ that has x held constant with $x = 1.75$. Write a couple of sentences that describe whether the slope of the tangent

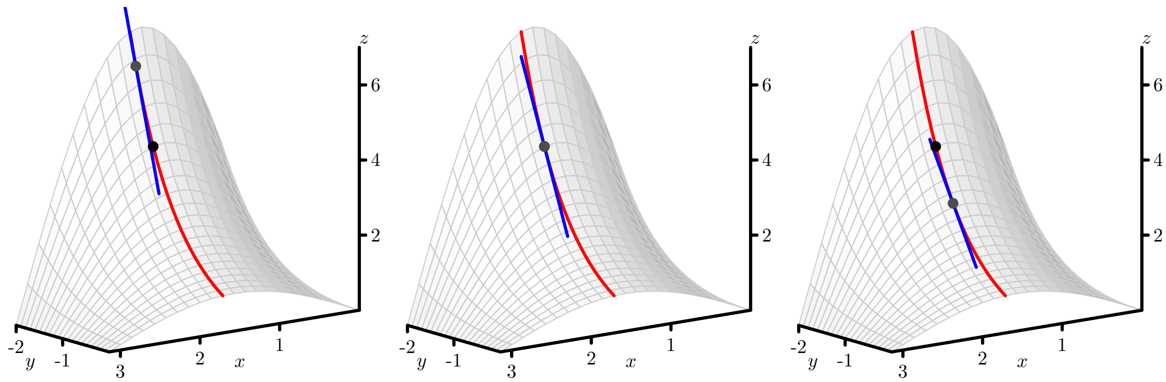

Figure 10.14: The tangent lines to a trace with increasing y.

lines to this curve increase or decrease as y increases, and, after computing $f_{yy}(x, y)$, explain how this observation is related to the value of $f_{yy}(1.75, y)$. Be sure to address the notion of concavity in your response.

(b) In Figure 10.15, we start to think about the mixed partial derivative, f_{xy}. Here, we first hold y constant to generate the first-order partial derivative f_x, and then we hold x constant to compute f_{xy}. This leads to first thinking about a trace with x being constant, followed by slopes of tangent lines in the y-direction that slide along the original trace. You might think of sliding your pencil down the trace with x constant in a way that its slope indicates $(f_x)_y$ in order to further animate the three snapshots shown in the figure. Based on Figure 10.15, is $f_{xy}(1.75, -1.5)$ positive or negative? Why?

(c) Determine the formula for $f_{xy}(x, y)$, and hence evaluate $f_{xy}(1.75, -1.5)$. How does this value compare with your observations in (b)?

(d) We know that $f_{xx}(1.75, -1.5)$ measures the concavity of the $y = -1.5$ trace, and that $f_{yy}(1.75, -1.5)$ measures the concavity of the $x = 1.75$ trace. What do you think $f_{xy}(1.75, -1.5)$ measures?

(e) On Figure 10.15, sketch the trace with $y = -1.5$, and sketch three tangent lines whose slopes correspond to the value of $f_{yx}(x, -1.5)$ for three different values of x, the middle of which is $x = -1.5$. Is $f_{yx}(1.75, -1.5)$ positive or negative? Why? What does $f_{yx}(1.75, -1.5)$ measure?

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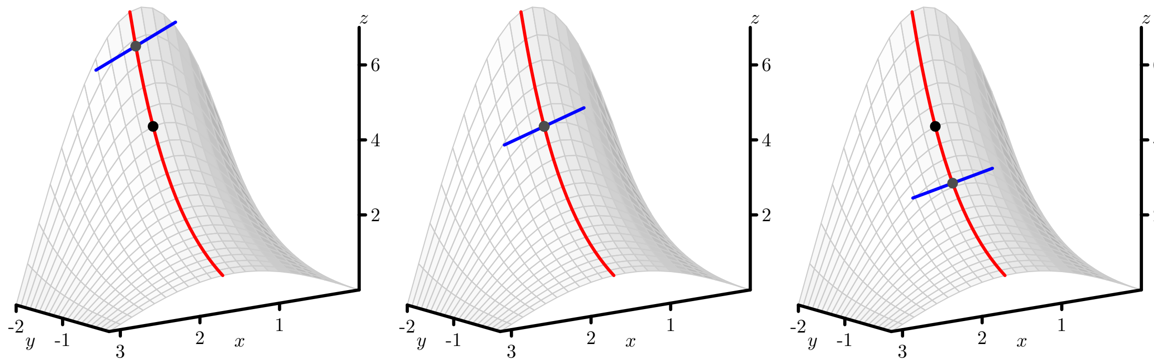

Figure 10.15: The trace of $z = f(x, y) = \sin(x)e^{-y}$ with $x = 1.75$, along with tangent lines in the y-direction at three different points.

Activity 10.10.

As we saw in Activity 10.6, the wind chill $w(v, T)$, in degrees Fahrenheit, is a function of the wind speed, in miles per hour, and the air temperature, in degrees Fahrenheit. Some values of the wind chill are recorded in Table 10.2.

$v \backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

Table 10.2: Wind chill as a function of wind speed and temperature.

(a) Estimate the partial derivatives $w_T(20, -15)$, $w_T(20, -10)$, and $w_T(20, -5)$. Use these results to estimate the second-order partial $w_{TT}(20, -10)$.

(b) In a similar way, estimate the second-order partial $w_{vv}(20, -10)$.

(c) Estimate the partial derivatives $w_T(20, -10)$, $w_T(25, -10)$, and $w_T(15, -10)$, and use your results to estimate the partial $w_{Tv}(20, -10)$.

(d) In a similar way, estimate the partial derivative $w_{vT}(20, -10)$.

(e) Write several sentences that explain what the values $w_{TT}(20, -10)$, $w_{vv}(20, -10)$, and $w_{Tv}(20, -10)$ indicate regarding the behavior of $w(v, T)$.

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10.4 Linearization: Tangent Planes and Differentials

Preview Activity 10.4. Let $f(x, y) = 6 - \frac{x^2}{2} - y^2$, and let $(x_0, y_0) = (1, 1)$.

(a) Evaluate the function $f(x, y) = 6 - \frac{x^2}{2} - y^2$ and its partial derivatives at (x_0, y_0); that is, find $f(1, 1)$, $f_x(1, 1)$, and $f_y(1, 1)$.

(b) We know one point on the tangent plane; namely, the z-value of the tangent plane agrees with the z-value on the graph of the function $f(x, y) = 6 - \frac{x^2}{2} - y^2$ at the point (x_0, y_0). Use this observation to determine z_0 in the expression $z = z_0 + a(x - x_0) + b(y - y_0)$.

(c) Sketch the traces of the function $f(x, y) = 6 - \frac{x^2}{2} - y^2$ for $y = y_0 = 1$ and $x = x_0 = 1$ below in Figure 10.16.

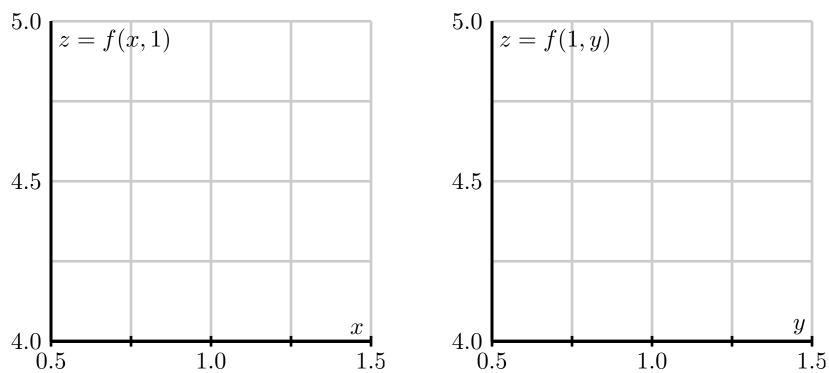

Figure 10.16: The traces of $f(x, y)$ with $y = y_0 = 1$ and $x = x_0 = 1$.

(d) Determine the equation of the tangent line of the trace with $y = 1$ at the point $x_0 = 1$.

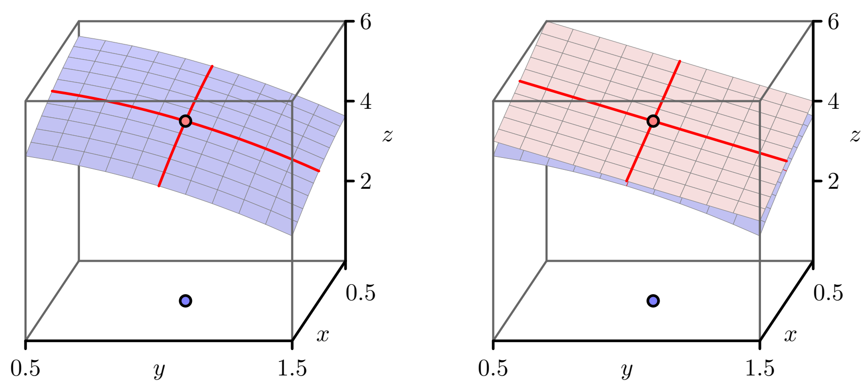

Figure 10.17: The traces of $f(x, y)$ and the tangent plane.

(e) Figure 10.17 shows the traces of the function and the traces of the tangent plane. Explain how the tangent line of the trace of f, whose equation you found in the last part of this activity, is related to the tangent plane. How does this observation help you determine the constant a in the expression for the tangent plane $z = z_0 + a(x - x_0) + b(y - y_0)$? (Hint: How do you think $f_x(x_0, y_0)$ should be related to $z_x(x_0, y_0)$?)

(f) In a similar way to what you did in (d), determine the equation of the tangent line of the trace with $x = 1$ at the point $y_0 = 1$. Explain how this tangent line is related to the tangent plane, and use this observation to determine the constant b in the expression for the tangent plane $z = z_0 + a(x - x_0) + b(y - y_0)$. (Hint: How do you think $f_y(x_0, y_0)$ should be related to $z_y(x_0, y_0)$?)

(g) Finally, write the equation $z = z_0 + a(x - x_0) + b(y - y_0)$ of the tangent plane to the graph of $f(x, y) = 6 - x^2/2 - y^2$ at the point $(x_0, y_0) = (1, 1)$.

$\triangleleft$

Activity 10.11.

Find the equation of the tangent plane to $f(x, y) = x^2y$ at the point $(1, 2)$.

$\triangleleft$

Activity 10.12.

In what follows, we find the linearization of several different functions that are given in algebraic, tabular, or graphical form.

(a) Find the linearization $L(x, y)$ for the function g defined by

$$g(x, y) = \frac{x}{x^2 + y^2}$$

at the point $(1, 2)$. Then use the linearization to estimate the value of $g(0.8, 2.3)$.

(b) Table 10.3 provides a collection of values of the wind chill $w(v, T)$, in degrees Fahrenheit, as a function of wind speed, in miles per hour, and temperature, also in degrees Fahrenheit.

$v\backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

Table 10.3: Wind chill as a function of wind speed and temperature.

Use the data to first estimate the appropriate partial derivatives, and then find the linearization $L(v, T)$ at the point $(25, -10)$. Finally, use the linearization to estimate $w(25, -12)$, $w(23, -10)$, and $w(23, -12)$.

(c) Figure 10.18 gives a contour plot of a differentiable function f.

After estimating appropriate partial derivatives, determine the linearization $L(x, y)$ at the point $(2, 1)$, and use it to estimate $f(2.2, 1)$, $f(2, 0.8)$, and $f(2.2, 0.8)$.

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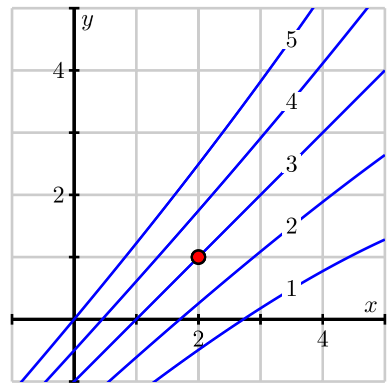

Figure 10.18: A contour plot of $f(x, y)$.

Activity 10.13.

The questions in this activity explore the differential in several different contexts.

(a) Suppose that the elevation of a landscape is given by the function h, where we additionally know that $h(3, 1) = 4.35$, $h_x(3, 1) = 0.27$, and $h_y(3, 1) = -0.19$. Assume that x and y are measured in miles in the easterly and northerly directions, respectively, from some base point $(0, 0)$.

Your GPS device says that you are currently at the point $(3, 1)$. However, you know that the coordinates are only accurate to within 0.2 units; that is, $dx = \Delta x = 0.2$ and $dy = \Delta y = 0.2$. Estimate the uncertainty in your elevation using differentials.

(b) The pressure, volume, and temperature of an ideal gas are related by the equation

$$P = P(T, V) = 8.31T/V,$$

where P is measured in kilopascals, V in liters, and T in kelvin. Find the pressure when the volume is 12 liters and the temperature is 310 K. Use differentials to estimate the change in the pressure when the volume increases to 12.3 liters and the temperature decreases to 305 K.

(c) Refer to Table 10.3, the table of values of the wind chill $w(v, T)$, in degrees Fahrenheit, as a function of temperature, also in degrees Fahrenheit, and wind speed, in miles per hour.

Suppose your anemometer says the wind is blowing at 25 miles per hour and your thermometer shows a reading of $-15°$ degrees. However, you know your thermometer is only accurate to within $2°$ degrees and your anemometer is only accurate to within 3 miles per hour. What is the wind chill based on your measurements? Estimate the uncertainty in your measurement of the wind chill.

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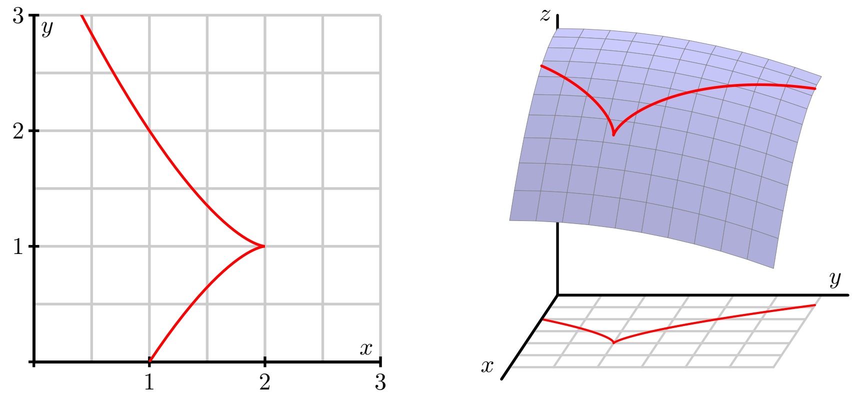

Figure 10.19: At left, your position in the plane; at right, the corresponding temperature.

10.5 The Chain Rule

Preview Activity 10.5. Suppose you are driving around in the x-y plane in such a way that your position at time t is given by the vector-valued function

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 2 - t^2, t^3 + 1\rangle.$$

The path taken is shown on the left of Figure 10.19.

Suppose, furthermore, that the temperature at a point in the plane is given by

$$T(x, y) = 10 - \frac{1}{2}x^2 - \frac{1}{5}y^2,$$

and note that the surface generated by T is shown on the right of Figure 10.19. Therefore, as time passes, your position $(x(t), y(t))$ changes, and, as your position changes, the temperature $T(x, y)$ also changes.

(a) The position function $\mathbf{r}$ provides a parameterization $x = x(t)$ and $y = y(t)$ of the position at time t. By substituting $x(t)$ for x and $y(t)$ for y in the formula for T, we can write $T = T(x(t), y(t))$ as a function of t. Make these substitutions to write T as a function of t and then use the Chain Rule from single variable calculus to find $\frac{dT}{dt}$. (Do not do any algebra to simplify the derivative, either before taking the derivative, nor after.)

(b) Now we want to understand how the result from part (a) can be obtained from T as a multivariable function. Recall from the previous section that small changes in x and y produce a change in T that is approximated by

$$\Delta T \approx T_x \Delta x + T_y \Delta y.$$

The Chain Rule tells us about the instantaneous rate of change of T, and this can be found as

$$\lim_{\Delta t \to 0} \frac{\Delta T}{\Delta t} = \lim_{\Delta t \to 0} \frac{T_x \Delta x + T_y \Delta y}{\Delta t}. \tag{10.1}$$

Use equation (10.1) to explain why the instantaneous rate of change of T that results from a change in t is

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}. \tag{10.2}$$

(c) Using the original formulas for T, x, and y in the problem statement, calculate all of the derivatives in Equation (10.2) and hence write the right-hand side of Equation (10.2) in terms of x, y, and t.

(d) Compare the results of parts (a) and (c). Write a couple of sentences that identify specifically how each term in (c) relates to a corresponding terms in (a). This connection between parts (a) and (c) provides a multivariable version of the Chain Rule.

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Activity 10.14.

In the following questions, we apply the recently-developed Chain Rule in several different contexts.

(a) Suppose that we have a function z defined by $z(x, y) = x^2 + xy^3$. In addition, suppose that x and y are restricted to points that move around the plane by following a circle of radius 2 centered at the origin that is parameterized by

$$x(t) = 2\cos t, \qquad \text{and} \qquad y(t) = 2\sin t.$$

Use the Chain Rule to find the resulting instantaneous rate of change $\frac{dz}{dt}$.

(b) Suppose that the temperature on a metal plate is given by the function T with

$$T(x, y) = 100 - (x^2 + 4y^2),$$

where the temperature is measured in degrees Fahrenheit and x and y are each measured in feet.

i. Find T_x and T_y. What are the units on these partial derivatives?

ii. Suppose an ant is walking along the x-axis at the rate of 2 feet per minute toward the origin. When the ant is at the point $(2, 0)$, what is the instantaneous rate of change in the temperature dT/dt that the ant experiences. Include units on your response.

iii. Suppose instead that the ant walks along an ellipse with $x = 6\cos(t)$ and $y = 3\sin(t)$, where t is measured in minutes. Find $\frac{dT}{dt}$ at $t = \pi/6$, $t = \pi/4$, and $t = \pi/3$. What does this seem to tell you about the path along which the ant is walking?

(c) Suppose that you are walking along a surface whose elevation is given by a function f. Furthermore, suppose that if you consider how your location corresponds to points in the x-y plane, you know that when you pass the point $(2, 1)$, your velocity vector is $\mathbf{v} = \langle -1, 2 \rangle$. If some contours of the function $f(x, y)$ are as shown in Figure 10.20, estimate the rate of change df/dt when you pass through $(2, 1)$.

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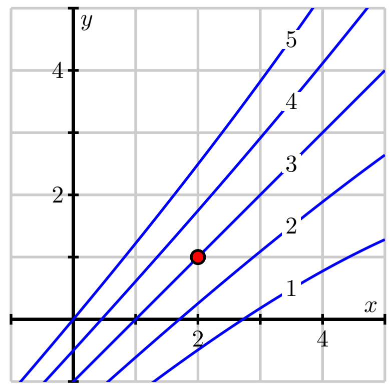

Figure 10.20: Some contours of $f(x, y)$.

Activity 10.15.

(a) Figure 10.21 shows the tree diagram we construct when (a) z depends on w, x, and y, (b) w, x, and y each depend on u and v, and (c) u and v depend on t.

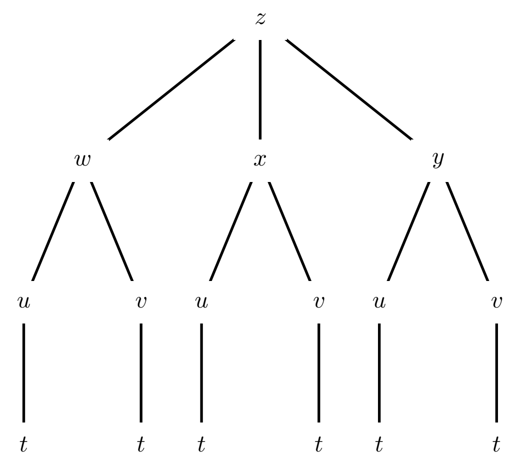

Figure 10.21: Three levels of dependencies

i. Label the edges with the appropriate derivatives.

ii. Use the Chain Rule to write $\dfrac{dz}{dt}$.

(b) Suppose that $z = x^2 - 2xy^2$ and that

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

i. Construct a tree diagram representing the dependencies of z on x and y and x and y on r and θ.

ii. Use the tree diagram to find $\frac{\partial z}{\partial r}$.

iii. Now suppose that $r = 3$ and $\theta = \pi/6$. Find the values of x and y that correspond to these given values of r and θ, and then use the Chain Rule to find the value of the partial derivative $\frac{\partial z}{\partial \theta}|_{(3,\frac{\pi}{6})}$.

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10.6 Directional Derivatives and the Gradient

Preview Activity 10.6. Let's consider the function f defined by

$$f(x, y) = 30 - x^2 - \frac{1}{2}y^2,$$

and suppose that f measures the temperature, in degrees Celsius, at a given point in the plane, where x and y are measured in feet. Assume that the positive x-axis points due east, while the positive y-axis points due north. A contour plot of f is shown in Figure 10.22

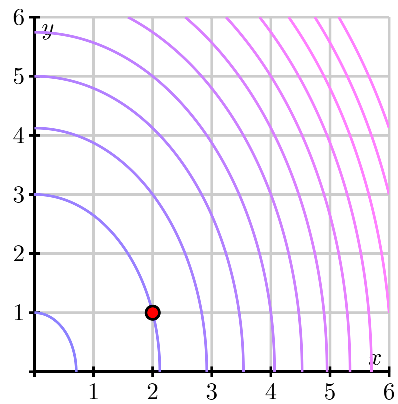

Figure 10.22: A contour plot of $f(x, y) = 30 - x^2 - \frac{1}{2}y^2$.

(a) Suppose that a person is walking due east, and thus parallel to the x-axis. At what instantaneous rate is the temperature changing at the moment she passes the point $(2, 1)$? What are the units on this rate of change?

(b) Next, determine the instantaneous rate of change of temperature at the point $(2, 1)$ if the person is instead walking due north. Again, include units on your result.

(c) Now, rather than walking due east or due north, let's suppose that the person is walking with velocity given by the vector $\mathbf{v} = \langle 3, 4 \rangle$, where time is measured in seconds. Note that the person's speed is thus $|\mathbf{v}| = 5$ feet per second.

Find parametric equations for the person's path; that is, parameterize the line through $(2, 1)$ using the direction vector $\mathbf{v} = \langle 3, 4 \rangle$. Let $x(t)$ denote the x-coordinate of the line, and $y(t)$ its y-coordinate.

(d) With the parameterization in (c), we can now view the temperature f as not only a function of x and y, but also of time, t. Hence, use the chain rule to determine the value of $\frac{df}{dt}|_{t=0}$. What are the units on your answer? What is the practical meaning of this result?

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Activity 10.16.

Let $f(x, y) = 3xy - x^2y^3$.

(a) Determine $f_x(x, y)$ and $f_y(x, y)$.

(b) Use Equation (**??**) to determine $D_{\mathbf{i}}f(x, y)$ and $D_{\mathbf{j}}f(x, y)$. What familiar function is $D_{\mathbf{i}}f(x, y)$? What familiar function is $D_{\mathbf{j}}f(x, y)$?

(c) Use Equation (**??**) to find the derivative of f in the direction of the vector $\mathbf{v} = \langle 2, 3\rangle$ at the point $(1, -1)$. Remember that a unit direction vector is needed.

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Activity 10.17.

Let's consider the function f defined by $f(x, y) = x^2 - y^2$. Some contours for this function are shown in Figure 10.23.

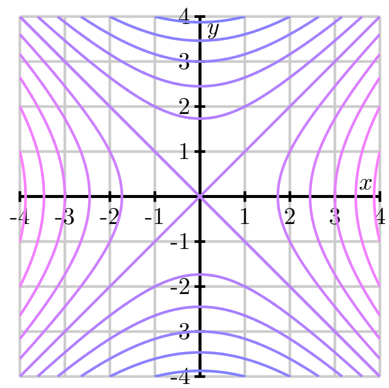

Figure 10.23: Contour lines of $f(x, y) = x^2 - y^2$.

(a) Find the gradient $\nabla f(x, y)$.

(b) For each of the following points (x_0, y_0), evaluate the gradient $\nabla f(x_0, y_0)$ and sketch the gradient vector with its tail at (x_0, y_0). Some of the vectors are too long to fit onto the plot, but we'd like to draw them to scale; to do so, scale each vector by a factor of $1/4$.

- $(x_0, y_0) = (2, 0)$
- $(x_0, y_0) = (0, 2)$
- $(x_0, y_0) = (2, 2)$
- $(x_0, y_0) = (2, 1)$
- $(x_0, y_0) = (-3, 2)$
- $(x_0, y_0) = (-2, -4)$
- $(x_0, y_0) = (0, 0)$

(c) What do you notice about the relationship between the gradient at (x_0, y_0) and the contour line passing through that point?

(d) Does f increase or decrease in the direction of $\nabla f(x_0, y_0)$? Provide a justification for your response.

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Activity 10.18.

In this activity we investigate how the gradient is related to the directions of greatest increase and decrease of a function. Let f be a differentiable function and $\mathbf{u}$ a unit vector.

(a) Let θ be the angle between $\nabla f(x_0, y_0)$ and $\mathbf{u}$. Explain why

$$D_{\mathbf{u}} f(x_0, y_0) = |\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle| \cos(\theta). \tag{10.3}$$

(b) At the point (x_0, y_0), the only quantity in Equation (10.3) that can change is θ (which determines the direction $\mathbf{u}$ of travel). Explain why $\theta = 0$ makes the quantity $|\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle| \cos(\theta)$ as large as possible.

(c) When $\theta = 0$, in what direction does the unit vector $\mathbf{u}$ point relative to $\nabla f(x_0, y_0)$? Why? What does this tell us about the direction of greatest increase of f at the point (x_0, y_0)?

(d) In what direction, relative to $\nabla f(x_0, y_0)$, does f decrease most rapidly at the point (x_0, y_0)?

(e) State the unit vectors $\mathbf{u}$ and $\mathbf{v}$ (in terms of $\nabla f(x_0, y_0)$) that provide the directions of greatest increase and decrease for the function f at the point (x_0, y_0). What important assumption must we make regarding $\nabla f(x_0, y_0)$ in order for these vectors to exist?

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Activity 10.19.

Consider the function f defined by $f(x, y) = 2x^2 - xy + 2y$.

(a) Find the gradient $\nabla f(1, 2)$ and sketch it on Figure 10.24.

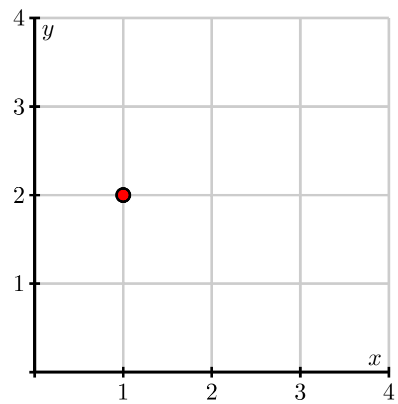

Figure 10.24: A plot for the gradient $\nabla f(1, 2)$.

(b) Sketch the unit vector $\mathbf{z} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ on Figure 10.24 with its tail at $(1, 2)$. Now find the directional derivative $D_{\mathbf{z}} f(1, 2)$.

(c) What is the slope of the graph of f in the direction $\mathbf{z}$? What does the sign of the directional derivative tell you?

(d) Consider the vector $\mathbf{v} = \langle 2, -1 \rangle$ and sketch $\mathbf{v}$ on Figure 10.24 with its tail at $(1, 2)$. Find a unit vector $\mathbf{w}$ pointing in the same direction of $\mathbf{v}$. Without computing $D_{\mathbf{w}} f(1, 2)$, what do you know about the sign of this directional derivative? Now verify your observation by computing $D_{\mathbf{w}} f(1, 2)$.

(e) In which direction (that is, for what unit vector $\mathbf{u}$) is $D_{\mathbf{u}} f(1, 2)$ the greatest? What is the slope of the graph in this direction?

(f) Corresponding, in which direction is $D_{\mathbf{u}} f(1, 2)$ least? What is the slope of the graph in this direction?

(g) Sketch two unit vectors $\mathbf{u}$ for which $D_{\mathbf{u}} f(1, 2) = 0$ and then find component representations of these vectors.

(h) Suppose you are standing at the point $(3, 3)$. In which direction should you move to cause f to increase as rapidly as possible? At what rate does f increase in this direction?

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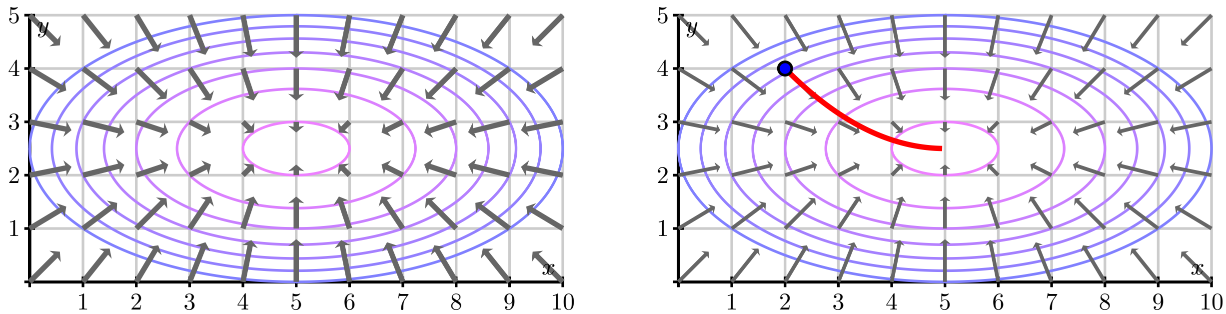

Figure 10.25: Contours and gradient for $T(x, y)$ and the missile's path.

Activity 10.20.

(a) The temperature $T(x, y)$ has its maximum value at the fighter jet's location. State the fighter jet's location and explain how Figure 10.25 tells you this.

(b) Determine ∇T at the fighter jet's location and give a justification for your response.

(c) Suppose that a different function f has a local maximum value at (x_0, y_0). Sketch the behavior of some possible contours near this point. What is $\nabla f(x_0, y_0)$?

(d) Suppose that a function g has a local minimum value at (x_0, y_0). Sketch the behavior of some possible contours near this point. What is $\nabla g(x_0, y_0)$?

(e) If a function g has a local minimum at (x_0, y_0), what is the direction of greatest increase of g at (x_0, y_0)?

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10.7 Optimization

Preview Activity 10.7. Let $z = f(x, y)$ be a differentiable function, and suppose that at the point (x_0, y_0), f achieves a local maximum. That is, the value of $f(x_0, y_0)$ is greater than the value of $f(x, y)$ for all (x, y) nearby (x_0, y_0). You might find it helpful to sketch a rough picture of a possible function f that has this property.

(a) If we consider the trace given by holding $y = y_0$ constant, then the single-variable function $f(x, y_0)$ must have a local maximum at x_0. What does this say about the value of the partial derivative $f_x(x_0, y_0)$?

(b) In the same way, the trace given by holding $x = x_0$ constant has a local maximum at $y = y_0$. What does this say about the value of the partial derivative $f_y(x_0, y_0)$?

(c) What may we now conclude about the gradient $\nabla f(x_0, y_0)$ at the local maximum? How is this consistent with the statement "f increases most rapidly in the direction $\nabla f(x_0, y_0)$?"

(d) How will the tangent plane to the surface $z = f(x, y)$ appear at the point $(x_0, y_0, f(x_0, y_0))$?

(e) By first computing the partial derivatives, find any points at which the function $f(x, y) = 2x - x^2 - (y + 2)^2$ may have a local maximum.

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Activity 10.21.

Find the critical points of each of the following functions. Then, using appropriate technology (e.g., Wolfram|Alpha or CalcPlot3D[1]), plot the graphs of the surfaces near each critical value and compare the graph to your work.

(a) $f(x, y) = 2 + x^2 + y^2$

(b) $f(x, y) = 2 + x^2 - y^2$

(c) $f(x, y) = 2x - x^2 - \frac{1}{4}y^2$

(d) $f(x, y) = |x| + |y|$

(e) $f(x, y) = 2xy - 4x + 2y - 3.$

$\triangleleft$

[1]at http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm

Activity 10.22.

Find the critical points of the following functions and use the Second Derivative Test to classify the critical points.

(a) $f(x,y) = 3x^3 + y^2 - 9x + 4y$

(b) $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

(c) $f(x,y) = x^3 + y^3 - 3xy$.

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Activity 10.23.

While the quantity of a product demanded by consumers is often a function of the price of the product, the demand for a product may also depend on the price of other products. For instance, the demand for blue jeans at Old Navy may be affected not only by the price of the jeans themselves, but also by the price of khakis.

Suppose we have two goods whose respective prices are p_1 and p_2. The demand for these goods, q_1 and q_2, depend on the prices as

$$q_1 = 150 - 2p_1 - p_2 \tag{10.6}$$

$$q_2 = 200 - p_1 - 3p_2. \tag{10.7}$$

The seller would like to set the prices p_1 and p_2 in order to maximize revenue. We will assume that the seller meets the full demand for each product. Thus, if we let R be the revenue obtained by selling q_1 items of the first good at price p_1 per item and q_2 items of the second good at price p_2 per item, we have

$$R = p_1q_1 + p_2q_2.$$

We can then write the revenue as a function of just the two variables p_1 and p_2 by using Equations (10.6) and (10.7), giving us

$$R(p_1, p_2) = p_1(150 - 2p_1 - p_2) + p_2(200 - p_1 - 3p_2) = 150p_1 + 200p_2 - 2p_1p_2 - 2p_1^2 - 3p_2^2.$$

A graph of R as a function of p_1 and p_2 is shown in Figure 10.26.

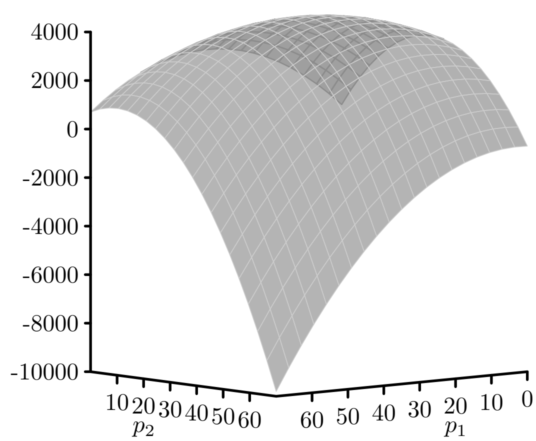

Figure 10.26: A revenue function.

(a) Find all critical points of the revenue function, $R(p_1, p_2)$.

(b) Apply the Second Derivative Test to determine the type of any critical points.

(c) Where should the seller set the prices p_1 and p_2 to maximize the revenue?

◁

Activity 10.24.

Let $f(x,y) = x^2 - 3y^2 - 4x + 6y$ with triangular domain R whose vertices are at $(0,0)$, $(4,0)$, and $(0,4)$. The domain R and a graph of f on the domain appear in Figure 10.27.

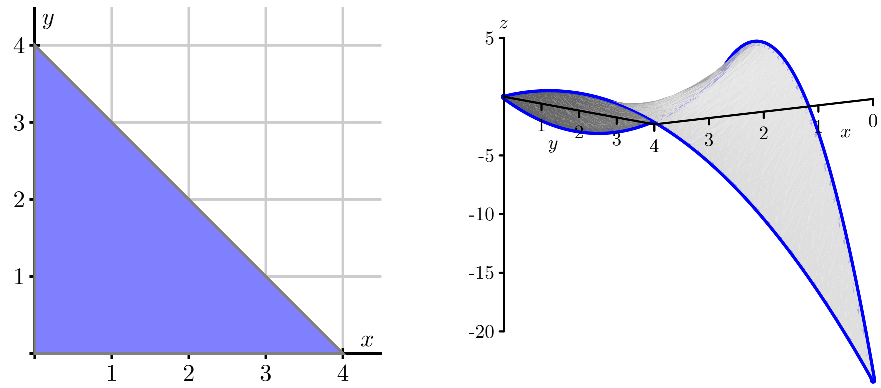

Figure 10.27: The domain of the function $f(x,y) = x^2 - 3y^2 - 4x + 6y$ and its graph.

(a) Find all of the critical points of f in R.

(b) Parameterize the horizontal leg of the triangular domain, and find the critical points of f on that leg.

(c) Parameterize the vertical leg of the triangular domain, and find the critical points of f on that leg.

(d) Parameterize the hypotenuse of the triangular domain, and find the critical points of f on the hypotenuse.

(e) Find the absolute maximum and absolute minimum value of f on R.

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10.8 Constrained Optimization:Lagrange Multipliers

Preview Activity 10.8. According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" we mean the perimeter of the smallest end. Our goal is to find the largest possible volume of a rectangular parcel with a square end that can be sent by mail.[2] If we let x be the length of the side of one square end of the package and y the length of the package, then we want to maximize the volume $f(x, y) = x^2y$ of the box subject to the constraint that the girth ($4x$) plus the length (y) is as large as possible, or $4x + y = 108$. The equation $4x + y = 108$ is thus an external constraint on the variables.

(a) The constraint equation involves the function g that is given by

$$g(x, y) = 4x + y.$$

Explain why the constraint is a contour of g, and is therefore a two-dimensional curve.

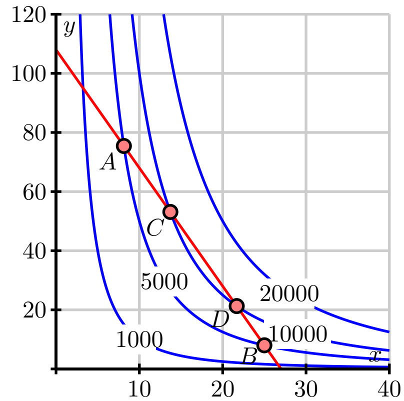

Figure 10.28: Contours of f and the constraint equation $g(x, y) = 108$.

(b) Figure 10.28 shows the graph of the constraint equation $g(x, y) = 108$ along with a few contours of the volume function f. Since our goal is to find the maximum value of f subject to the constraint $g(x, y) = 108$, we want to find the point on our constraint curve that intersects the contours of f at which f has its largest value.

i. Points A and B in Figure 10.28 lie on a contour of f and on the constraint equation $g(x, y) = 108$. Explain why neither A nor B provides a maximum value of f that satisfies the constraint.

ii. Points C and D in Figure 10.28 lie on a contour of f and on the constraint equation $g(x, y) = 108$. Explain why neither C nor D provides a maximum value of f that satisfies the constraint.

[2] We solved this applied optimization problem in single variable *Active Calculus*, so it may look familiar. We take a different approach in this section, and this approach allows us to view most applied optimization problems from single variable calculus as constrained optimization problems, as well as provide us tools to solve a greater variety of optimization problems.

iii. Based on your responses to parts i. and ii., draw the contour of f on which you believe f will achieve a maximum value subject to the constraint $g(x, y) = 108$. Explain why you drew the contour you did.

(c) Recall that $g(x, y) = 108$ is a contour of the function g, and that the gradient of a function is always orthogonal to its contours. With this in mind, how should ∇f and ∇g be related at the optimal point? Explain.

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Activity 10.25.

A cylindrical soda can holds about 355 cc of liquid. In this activity, we want to find the dimensions of such a can that will minimize the surface area.

(a) What are the variables in this problem? What restriction(s), if any, are there on these variables?

(b) What quantity do we want to optimize in this problem? What equation describes the constraint?

(c) Find λ and the values of your variables that satisfy Equation (**??**) in the context of this problem.

(d) Determine the dimensions of the pop can that give the desired solution to this constrained optimization problem.

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Activity 10.26.

Use the method of Lagrange multipliers to find the dimensions of the least expensive packing crate with a volume of 240 cubic feet when the material for the top costs \$2 per square foot, the bottom is \$3 per square foot and the sides are \$1.50 per square foot.

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Chapter 11

Multiple Integrals

11.1 Double Riemann Sums and Double Integrals over Rectangles

Preview Activity 11.1. In this activity we introduce the concept of a double Riemann sum.

(a) Review the concept of the Riemann sum from single-variable calculus. Then, explain how we define the definite integral $\int_a^b f(x)\, dx$ of a continuous function of a single variable x on an interval $[a, b]$. Include a sketch of a continuous function on an interval $[a, b]$ with appropriate labeling in order to illustrate your definition.

(b) In our upcoming study of integral calculus for multivariable functions, we will first extend the idea of the single-variable definite integral to functions of two variables over rectangular domains. To do so, we will need to understand how to partition a rectangle into subrectangles. Let R be rectangular domain $R = \{(x, y) : 0 \le x \le 6, 2 \le y \le 4\}$ (we can also represent this domain with the notation $[0, 6] \times [2, 4]$), as pictured in Figure 11.1.

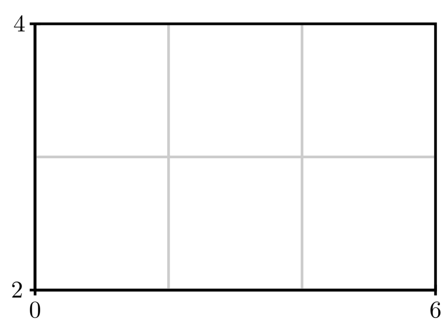

Figure 11.1: Rectangular domain R with subrectangles.

To form a partition of the full rectangular region, R, we will partition both intervals $[0, 6]$ and $[2, 4]$; in particular, we choose to partition the interval $[0, 6]$ into three uniformly sized subintervals and the interval $[2, 4]$ into two evenly sized subintervals as shown in Figure 11.1. In the following questions, we discuss how to identify the endpoints of each subinterval and the resulting subrectangles.

i. Let $0 = x_0 < x_1 < x_2 < x_3 = 6$ be the endpoints of the subintervals of $[0, 6]$ after partitioning. What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 3?

ii. Explicitly identify x_0, x_1, x_2, and x_3. On Figure 11.1 or your own version of the diagram, label these endpoints.

iii. Let $2 = y_0 < y_1 < y_2 = 4$ be the endpoints of the subintervals of $[2, 4]$ after partitioning. What is the length Δy of each subinterval $[y_{j-1}, y_j]$ for j from 1 to 2? Identify y_0, y_1, and y_2 and label these endpoints on Figure 11.1.

iv. Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Appropriately label each subrectangle in your drawing of Figure 11.1. How does the total number of subrectangles depend on the partitions of the intervals $[0, 6]$ and $[2, 4]$?

v. What is area ΔA of each subrectangle?

$\bowtie$

Activity 11.1.

Let $f(x, y) = 100 - x^2 - y^2$ be defined on the rectangular domain $R = [a, b] \times [c, d]$. Partition the interval $[a, b]$ into four uniformly sized subintervals and the interval $[c, d]$ into three evenly sized subintervals as shown in Figure 11.2. As we did in Preview Activity 11.1, we will need a method for identifying the endpoints of each subinterval and the resulting subrectangles.

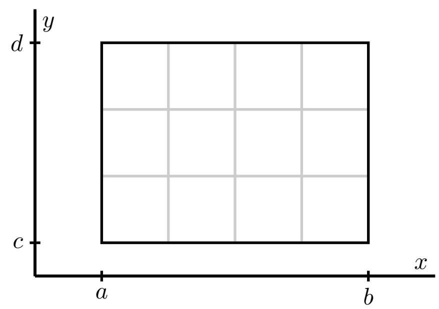

Figure 11.2: Rectangular domain with subrectangles.

(a) Let $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$ be the endpoints of the subintervals of $[a, b]$ after partitioning. Label these endpoints in Figure 11.2.

(b) What is the length Δx of each subinterval $[x_{i-1}, x_i]$? Your answer should be in terms of a and b.

(c) Let $c = y_0 < y_1 < y_2 < y_3 = d$ be the endpoints of the subintervals of $[c, d]$ after partitioning. Label these endpoints in Figure 11.2.

(d) What is the length Δy of each subinterval $[y_{j-1}, y_j]$? Your answer should be in terms of c and d.

(e) The partitions of the intervals $[a, b]$ and $[c, d]$ partition the rectangle R into subrectangles. How many subrectangles are there?

(f) Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Label each subrectangle in Figure 11.2.

(g) What is area ΔA of each subrectangle?

(h) Now let $[a, b] = [0, 8]$ and $[c, d] = [2, 6]$. Let (x^*_{11}, y^*_{11}) be the point in the upper right corner of the subrectangle R_{11}. Identify and correctly label this point in Figure 11.2. Calculate the product

$$f(x^*_{11}, y^*_{11})\Delta A.$$

Explain, geometrically, what this product represents.

(i) For each i and j, choose a point (x^*_{ij}, y^*_{ij}) in the subrectangle $R_{i,j}$. Identify and correctly label these points in Figure 11.2. Explain what the product

$$f(x^*_{ij}, y^*_{ij})\Delta A$$

represents.

(j) If we were to add all the values $f(x_{ij}^*, y_{ij}^*)\Delta A$ for each i and j, what does the resulting number approximate about the surface defined by f on the domain R? (You don't actually need to add these values.)

(k) Write a double sum using summation notation that expresses the arbitrary sum from part (j).

◁

Activity 11.2.

Let $f(x, y) = x + 2y$ and let $R = [0, 2] \times [1, 3]$.

(a) Draw a picture of R. Partition $[0, 2]$ into 2 subintervals of equal length and the interval $[1, 3]$ into two subintervals of equal length. Draw these partitions on your picture of R and label the resulting subrectangles using the labeling scheme we established in the definition of a double Riemann sum.

(b) For each i and j, let (x^*_{ij}, y^*_{ij}) be the midpoint of the rectangle R_{ij}. Identify the coordinates of each (x^*_{ij}, y^*_{ij}). Draw these points on your picture of R.

(c) Calculate the Riemann sum

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A$$

using the partitions we have described. If we let (x^*_{ij}, y^*_{ij}) be the midpoint of the rectangle R_{ij} for each i and j, then the resulting Riemann sum is called a *midpoint sum*.

(d) Give two interpretations for the meaning of the sum you just calculated.

◁

Activity 11.3.

Let $f(x, y) = \sqrt{4 - y^2}$ on the rectangular domain $R = [1, 7] \times [-2, 2]$. Partition $[1, 7]$ into 3 equal length subintervals and $[-2, 2]$ into 2 equal length subintervals. A table of values of f at some points in R is given in Table 11.1, and a graph of f with the indicated partitions is shown in Figure 11.3.

	-2	-1	0	1	2
1	0	$\sqrt{3}$	2	$\sqrt{3}$	0
2	0	$\sqrt{3}$	2	$\sqrt{3}$	0
3	0	$\sqrt{3}$	2	$\sqrt{3}$	0
4	0	$\sqrt{3}$	2	$\sqrt{3}$	0
5	0	$\sqrt{3}$	2	$\sqrt{3}$	0
6	0	$\sqrt{3}$	2	$\sqrt{3}$	0
7	0	$\sqrt{3}$	2	$\sqrt{3}$	0

Table 11.1: Table of values of $f(x, y) = \sqrt{4 - y^2}$.

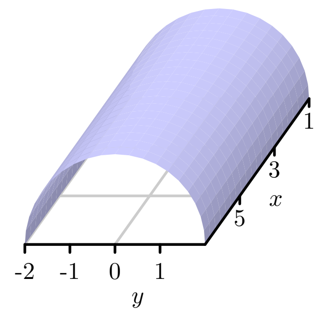

Figure 11.3: Graph of $f(x, y) = \sqrt{4 - y^2}$ on R.

(a) Outline the partition of R into subrectangles on the table of values in Table 11.1.

(b) Calculate the double Riemann sum using the given partition of R and the values of f in the upper right corner of each subrectangle.

(c) Use geometry to calculate the exact value of $\iint_R f(x, y)\, dA$ and compare it to your approximation. How could we obtain a better approximation?

$\triangleleft$

11.2 Iterated Integrals

Preview Activity 11.2. Let $f(x, y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3, 3] \times [-4, 4]$.

As with partial derivatives, we may treat one of the variables in f as constant and think of the resulting function as a function of a single variable. Now we investigate what happens if we integrate instead of differentiate.

(a) Choose a fixed value of x in the interior of $[-3, 3]$. Let

$$A(x) = \int_{-4}^{4} f(x, y)\, dy.$$

What is the geometric meaning of the value of $A(x)$ relative to the surface defined by f. (Hint: Think about the trace determined by the fixed value of x, and consider how $A(x)$ is related to Figure 11.4.)

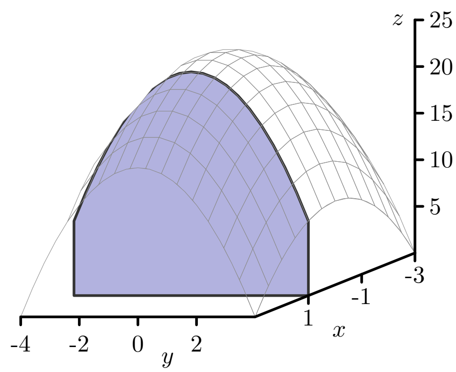

Figure 11.4: A cross section with fixed x.

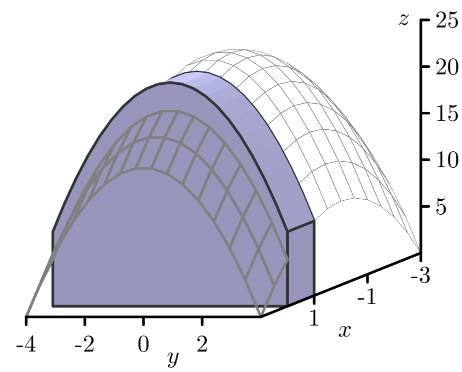

Figure 11.5: A cross section with fixed x and Δx.

(b) For a fixed value of x, say x_i^*, what is the geometric meaning of $A(x_i^*)\,\Delta x$? (Hint: Consider how $A(x_i^*)\Delta x$ is related to Figure 11.5.)

(c) Since f is continuous on R, we can define the function $A = A(x)$ at every value of x in $[-3, 3]$. Now think about subdividing the x-interval $[-3, 3]$ into m subintervals, and choosing a value x_i^* in each of those subintervals. What will be the meaning of the sum $\displaystyle\sum_{i=1}^{m} A(x_i^*)\,\Delta x$?

(d) Explain why $\int_{-3}^{3} A(x)\,dx$ will determine the exact value of the volume under the surface $z = f(x, y)$ over the rectangle R.

$\bowtie$

Activity 11.4.

Let $f(x, y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3, 3] \times [-4, 4]$.

(a) Viewing x as a fixed constant, use the Fundamental Theorem of Calculus to evaluate the integral

$$A(x) = \int_{-4}^{4} f(x, y)\, dy.$$

Note that you will be integrating with respect to y, and holding x constant. Your result should be a function of x only.

(b) Next, use your result from (a) along with the Fundamental Theorem of Calculus to determine the value of $\int_{-3}^{3} A(x)\, dx$.

(c) What is the value of $\iint_R f(x, y)\, dA$? What are two different ways we may interpret the meaning of this value?

◁

Activity 11.5.

Let $f(x,y) = x + y^2$ on the rectangle $R = [0,2] \times [0,3]$.

(a) Evaluate $\iint_R f(x,y)\,dA$ using an iterated integral. Choose an order for integration by deciding whether you want to integrate first with respect to x or y.

(b) Evaluate $\iint_R f(x,y)\,dA$ using the iterated integral whose order of integration is the opposite of the order you chose in (a).

$\triangleleft$

11.3 Double Integrals over General Regions

Preview Activity 11.3. A tetrahedron is a three-dimensional figure with four faces, each of which is a triangle. A picture of the tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is shown in Figure 11.6. If we place one vertex at the origin and let vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be determined by the edges of the tetrahedron that have one end at the origin, then a formula that tells us the volume V of the tetrahedron is

$$V = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{11.1}$$

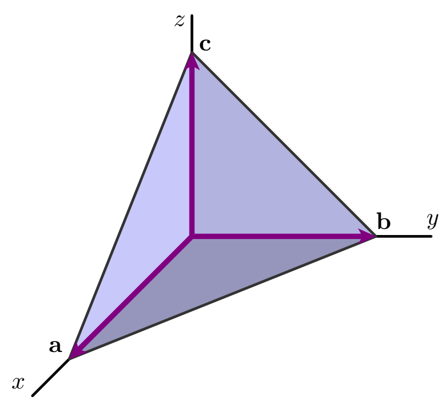

Figure 11.6: The tetrahedron T.

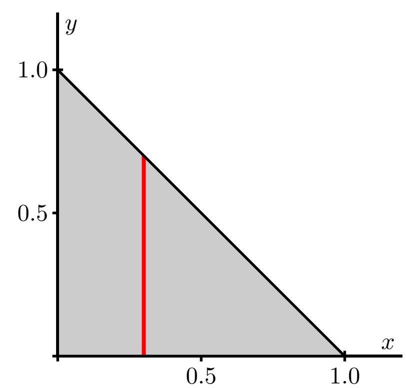

Figure 11.7: Projecting T onto the xy-plane.

(a) Use the formula (11.1) to find the volume of the tetrahedron T.

(b) Instead of memorizing or looking up the formula for the volume of a tetrahedron, we can use a double integral to calculate the volume of the tetrahedron T. To see how, notice that the top face of the tetrahedron T is the plane whose equation is

$$z = 1 - (x + y).$$

Provided that we can use an iterated integral on a non-rectangular region, the volume of the tetrahedron will be given by an iterated integral of the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} 1 - (x + y)\, dy\, dx.$$

The issue that is new here is how we find the limits on the integrals; note that the outer integral's limits are in x, while the inner ones are in y, since we have chosen $dA = dy\, dx$.

To see the domain over which we need to integrate, think of standing way above the tetrahedron looking straight down on it, which means we are projecting the entire tetrahedron onto the xy-plane. The resulting domain is the triangular region shown in Figure 11.7.

Explain why we can represent the triangular region with the inequalities

$$0 \leq y \leq 1 - x \quad \text{and} \quad 0 \leq x \leq 1.$$

(Hint: Consider the cross sectional slice shown in Figure 11.7.)

(c) Explain why it makes sense to now write the volume integral in the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} 1 - (x + y)\, dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 1 - (x + y)\, dy\, dx.$$

(d) Use the Fundamental Theorem of Calculus to evaluate the iterated integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 1 - (x + y)\, dy\, dx$$

and compare to your result from part (a). (As with iterated integrals over rectangular regions, start with the inner integral.)

$\bowtie$

Activity 11.6.

Consider the double integral $\iint_D (4 - x - 2y)\, dA$, where D is the triangular region with vertices (0,0), (4,0), and (0,2).

(a) Write the given integral as an iterated integral of the form $\iint_D (4 - x - 2y)\, dy\, dx$. Draw a labeled picture of D with relevant cross sections.

(b) Write the given integral as an iterated integral of the form $\iint_D (4 - x - 2y)\, dx\, dy$. Draw a labeled picture of D with relevant cross sections.

(c) Evaluate the two iterated integrals from (a) and (b), and verify that they produce the same value. Give at least one interpretation of the meaning of your result.

$\triangleleft$

Activity 11.7.

Consider the iterated integral $\int_{x=3}^{x=5} \int_{y=-x}^{y=x^2} (4x + 10y)\, dy\, dx$.

(a) Sketch the region of integration, D, for which

$$\iint_D (4x + 10y)\, dA = \int_{x=3}^{x=5} \int_{y=-x}^{y=x^2} (4x + 10y)\, dy\, dx.$$

(b) Determine the equivalent iterated integral that results from integrating in the opposite order ($dx\, dy$, instead of $dy\, dx$). That is, determine the limits of integration for which

$$\iint_D (4x + 10y)\, dA = \int_{y=?}^{y=?} \int_{x=?}^{x=?} (4x + 10y)\, dx\, dy.$$

(c) Evaluate one of the two iterated integrals above. Explain what the value you obtained tells you.

(d) Set up and evaluate a single definite integral to determine the exact area of D, $A(D)$.

(e) Determine the exact average value of $f(x, y) = 4x + 10y$ over D.

◁

Activity 11.8.

Consider the iterated integral $\displaystyle\int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\, dy\, dx$.

(a) Explain why we cannot antidifferentiate e^{y^2} with respect to y, and thus are unable to evaluate the iterated integral $\displaystyle\int_{x=0}^{x=4} \int_{y=0}^{y=x/2} e^{y^2}\, dy\, dx$ using the Fundamental Theorem of Calculus.

(b) Sketch the region of integration, D, so that $\displaystyle\iint_D e^{y^2}\, dA = \int_{x=0}^{x=4} \int_{y=0}^{y=x/2} e^{y^2}\, dy\, dx$.

(c) Rewrite the given iterated integral in the opposite order, using $dA = dx\, dy$.

(d) Use the Fundamental Theorem of Calculus to evaluate the iterated integral you developed in (d). Write one sentence to explain the meaning of the value you found.

(e) What is the important lesson this activity offers regarding the order in which we set up an iterated integral?

◁

11.4 Applications of Double Integrals

Preview Activity 11.4. Suppose that we have a flat, thin object (called a *lamina*) whose density varies across the object. We can think of the density on a lamina as a measure of mass per unit area. As an example, consider a circular plate D of radius 1 cm whose density δ varies depending on the distance from its center so that the density in grams per square centimeter at point (x, y) is

$$\delta(x, y) = 10 - 2(x^2 + y^2).$$

(a) Suppose that we partition the plate into subrectangles R_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$, of equal area ΔA, and select a point (x_{ij}^*, y_{ij}^*) in R_{ij} for each i and j.

What is the meaning of the quantity $\delta(x_{ij}^*, y_{ij}^*)\Delta A$?

(b) State a double Riemann sum that provides an approximation of the mass of the plate.

(c) Explain why the double integral

$$\iint_D \delta(x, y)\, dA$$

tells us the exact mass of the plate.

(d) Determine an iterated integral which, if evaluated, would give the exact mass of the plate. Do not actually evaluate the integral.[1]

$\bowtie$

[1] This integral is considerably easier to evaluate in polar coordinates, which we will learn more about in Section 11.5.

Activity 11.9.

Let D be a half-disk lamina of radius 3 in quadrants IV and I, centered at the origin as shown in Figure 11.8. Assume the density at point (x, y) is given by $\delta(x, y) = x$. Find the exact mass of the lamina.

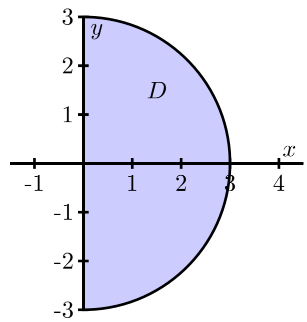

Figure 11.8: A half disk lamina.

◁

Activity 11.10.

Suppose we want to find the area of the bounded region D between the curves

$$y = 1 - x^2 \quad \text{and} \quad y = x - 1.$$

A picture of this region is shown in Figure 11.9.

(a) We know that the volume of a solid with constant height is given by the area of the base times the height. Hence, we may interpret the area of the region D as the volume of a solid with base D and of uniform height 1. Determine a double integral whose value is the area of D.

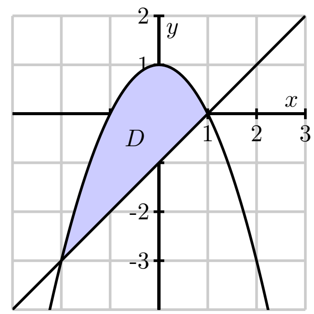

Figure 11.9: The graphs of $y = 1 - x^2$ and $y = x - 1$.

(b) Write an iterated integral whose value equals the double integral you found in (a).

(c) Use the Fundamental Theorem of Calculus to evaluate *only* the inner integral in the iterated integral in (b).

(d) After completing part (c), you should see a standard single area integral from calc II. Evaluate this remaining integral to find the exact area of D.

◁

Activity 11.11.

In this activity we determine integrals that represent the center of mass of a lamina D described by the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant if the density at point (x, y) is $\delta(x, y) = 6x + 6y + 6$. A picture of the lamina is shown in Figure 11.10.

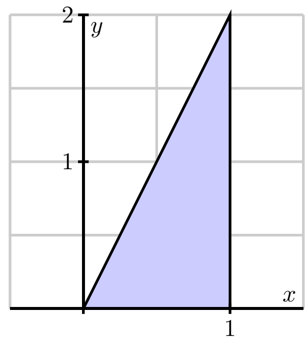

Figure 11.10: The lamina bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant.

(a) Set up an iterated integral that represents the mass of the lamina.

(b) Assume the mass of the lamina is 14. Set up two iterated integrals that represent the coordinates of the center of mass of the lamina.

$\triangleleft$

Activity 11.12.

A firm manufactures smoke detectors. Two components for the detectors come from different suppliers – one in Michigan and one in Ohio. The company studies these components for their reliability and their data suggests that if x is the life span (in years) of a randomly chosen component from the Michigan supplier and y the life span (in years) of a randomly chosen component from the Ohio supplier, then the joint probability density function f might be given by

$$f(x, y) = e^{-x}e^{-y}.$$

(a) Theoretically, the components might last forever, so the domain D of the function f is the set D of all (x, y) such that $x \geq 0$ and $y \geq 0$. To show that f is a probability density function on D we need to demonstrate that

$$\int\int_D f(x, y)\, dA = 1,$$

or that

$$\int_0^{\infty}\int_0^{\infty} f(x, y)\, dy\, dx = 1.$$

Use your knowledge of improper integrals to verify that f is indeed a probability density function.

(b) Assume that the smoke detector fails only if both of the supplied components fail. To determine the probability that a randomly selected detector will fail within one year, we will need to determine the probability that the life span of each component is between 0 and 1 years. Set up an appropriate iterated integral, and evaluate the integral to determine the probability.

(c) What is the probability that a randomly chosen smoke detector will fail between years 3 and 7?

(d) Suppose that the manufacturer determines that one of the components is more likely to fail than the other, and hence conjectures that the probability density function is instead $f(x, y) = Ke^{-x}e^{-2y}$. What is the value of K?

$\lhd$

11.5 Double Integrals in Polar Coordinates

Preview Activity 11.5. The coordinates of a point determine its location. In particular, the rectangular coordinates of a point P are given by an ordered pair (x, y), where x is the (signed) distance the point lies from the y-axis to P and y is the (signed) distance the point lies from the x-axis to P. In polar coordinates, we locate the point by considering the distance the point lies from the origin, $(0, 0)$, and the angle the line segment from the origin to P forms with the positive x-axis.

(a) Determine the rectangular coordinates of the following points:

i. The point P that lies 1 unit from the origin on the positive x-axis.

ii. The point Q that lies 2 units from the origin and such that $\overline{OQ}$ makes an angle of $\frac{\pi}{2}$ with the positive x-axis, where O is the origin, $(0, 0)$.

iii. The point R that lies 3 units from the origin such that $\overline{OR}$ makes an angle of $\frac{2\pi}{3}$ with the positive x-axis.

(b) Part (a) indicates that the two pieces of information completely determine the location of a point: either the traditional (x, y) coordinates, or alternately, the distance r from the point to the origin along with the angle θ that the line through the origin and the point makes with the positive x-axis. We write "(r, θ)" to denote the point's location in its polar coordinate representation. Find polar coordinates for the points with the given rectangular coordinates.

i. $(0, -1)$ ii. $(-2, 0)$ iii. $(-1, 1)$

(c) For each of the following points whose coordinates are given in polar form, determine the rectangular coordinates of the point.

i. $(5, \frac{\pi}{4})$ ii. $(2, \frac{5\pi}{6})$ iii. $(\sqrt{3}, \frac{5\pi}{3})$

$\bowtie$

Activity 11.13.

Most polar graphing devices[2] can plot curves in polar coordinates of the form $r = f(\theta)$. Use such a device to complete this activity.

(a) Before plotting the polar curve $r = 1$, think about what shape it should have, in light of how r is connected to x and y. Then use appropriate technology to draw the graph and test your intuition.

(b) The equation $\theta = 1$ does not define r as a function of θ, so we can't graph this equation on many polar plotters. What do you think the graph of the polar curve $\theta = 1$ looks like? Why?

(c) Before plotting the polar curve $r = \theta$, what do you think the graph looks like? Why? Use technology to plot the curve and compare your intuition.

(d) What about the curve $r = \sin(\theta)$? After plotting this curve, experiment with others of your choosing and think about why the curves look the way they do.

$\triangleleft$

[2]You can use your calculator in POL mode, or a web applet such as `http://webspace.ship.edu/msrenault/ggb/polar_grapher.html`

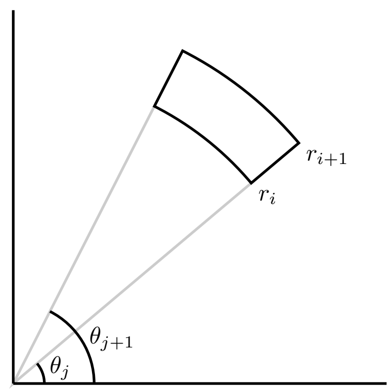

Figure 11.11: A polar rectangle.

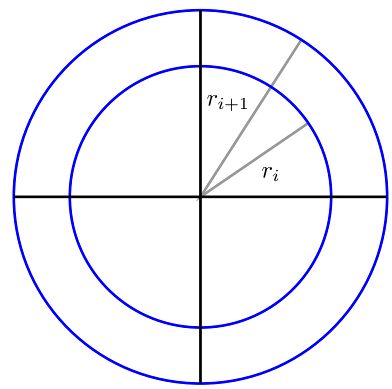

Figure 11.12: An annulus.

Activity 11.14.

Consider a polar rectangle R, with r between r_i and r_{i+1} and θ between θ_j and θ_{j+1} as shown in Figure 11.11. Let $\Delta r = r_{i+1} - r_i$ and $\Delta\theta = \theta_{j+1} - \theta_j$. Let ΔA be the area of this region.

(a) Explain why the area ΔA in polar coordinates is not $\Delta r\, \Delta\theta$.

(b) Now find ΔA by the following steps:

i. Find the area of the annulus (the washer-like region) between r_i and r_{i+1}, as shown at right in Figure 11.12. This area will be in terms of r_i and r_{i+1}.

ii. Observe that the region R is only a portion of the annulus, so the area ΔA of R is only a fraction of the area of the annulus. For instance, if $\theta_{i+1} - \theta_i$ were $\frac{\pi}{4}$, then the resulting wedge would be

$$\frac{\frac{\pi}{4}}{2\pi} = \frac{1}{4}$$

of the entire annulus. In this more general context, using the wedge between the two noted angles, what fraction of the area of the annulus is the area ΔA?

iii. Write an expression for ΔA in terms of r_i, r_{i+1}, θ_j, and θ_{j+1}.

iv. Finally, write the area ΔA in terms of r_i, r_{i+1}, Δr, and $\Delta\theta$, where each quantity appears only once in the expression. (Hint: Think about how to factor a difference of squares.)

(c) As we take the limit as Δr and $\Delta\theta$ go to 0, Δr becomes dr, $\Delta\theta$ becomes $d\theta$, and ΔA becomes dA, the area element. Using your work in (iv), write dA in terms of r, dr, and $d\theta$.

$\triangleleft$

Activity 11.15.

Let $f(x, y) = x + y$ and $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

(a) Write the double integral of f over D as an iterated integral in rectangular coordinates.

(b) Write the double integral of f over D as an iterated integral in polar coordinates.

(c) Evaluate one of the iterated integrals. Why is the final value you found not surprising?

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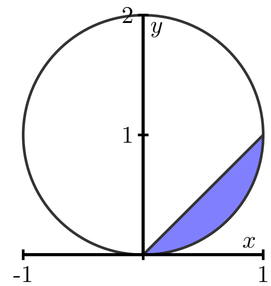

Figure 11.13: The graphs of $y = x$ and $x^2 + (y-1)^2 = 1$, for use in Activity 11.16.

Activity 11.16.

Consider the circle given by $x^2 + (y-1)^2 = 1$ as shown in Figure 11.13.

(a) Determine a polar curve in the form $r = f(\theta)$ that traces out the circle $x^2 + (y-1)^2 = 1$.

(b) Find the exact average value of $g(x, y) = \sqrt{x^2 + y^2}$ over the interior of the circle $x^2 + (y-1)^2 = 1$.

(c) Find the volume under the surface $h(x, y) = x$ over the region D, where D is the region bounded above by the line $y = x$ and below by the circle.

(d) Explain why in both (b) and (c) it is advantageous to use polar coordinates.

$\triangleleft$

11.6 Surfaces Defined Parametrically and Surface Area

Preview Activity 11.6. Recall the standard parameterization of the unit circle that is given by

$$x(t) = \cos(t) \quad \text{and} \quad y(t) = \sin(t),$$

where $0 \leq t \leq 2\pi$.

(a) Determine a parameterization of the circle of radius 1 in $\mathbb{R}^3$ that has its center at $(0, 0, 1)$ and lies in the plane $z = 1$.

(b) Determine a parameterization of the circle of radius 1 in 3-space that has its center at $(0, 0, -1)$ and lies in the plane $z = -1$.

(c) Determine a parameterization of the circle of radius 1 in 3-space that has its center at $(0, 0, 5)$ and lies in the plane $z = 5$.

(d) Taking into account your responses in (a), (b), and (c), describe the graph that results from the set of parametric equations

$$x(s, t) = \cos(t), \quad y(s, t) = \sin(t), \quad \text{and} \quad z(s, t) = s,$$

where $0 \leq t \leq 2\pi$ and $-5 \leq s \leq 5$. Explain your thinking.

(e) Just as a cylinder can be viewed as a "stack" of circles of constant radius, a cone can be viewed as a stack of circles with varying radius. Modify the parametrizations of the circles above in order to construct the parameterization of a cone whose vertex lies at the origin, whose base radius is 4, and whose height is 3, where the base of the cone lies in the plane $z = 3$. Use appropriate technology[3] to plot the parametric equations you develop. (Hint: The cross sections parallel to the xz plane are circles, with the radii varying linearly as z increases.)

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[3] e.g., http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

Activity 11.17.

In this activity, we seek a parametrization of the sphere of radius R centered at the origin, as shown on the left in Figure 11.14. Notice that this sphere may be obtained by revolving a half-circle contained in the xz-plane about the z-axis, as shown on the right.

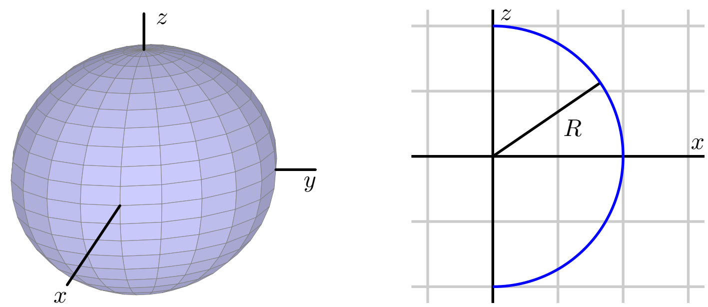

Figure 11.14: A sphere obtained by revolving a half-circle.

(a) Begin by writing a parametrization of this half-circle using the parameter s:

$$x(s) = \dots, \qquad z(s) = \dots.$$

Be sure to state the domain of the parameter s.

(b) By revolving the points on this half-circle about the z-axis, obtain a parametrization $\mathbf{r}(s,t)$ of the points on the sphere of radius R. Be sure to include the domain of both parameters s and t. (Hint: What is the radius of the circle obtained when revolving a point on the half-circle around the z axis?)

(c) Draw the surface defined by your parameterization with appropriate technology[4].

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[4]e.g., http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm or http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

Activity 11.18.

Consider the cylinder with radius a and height h defined parametrically by

$$\mathbf{r}(s,t) = a\cos(s)\mathbf{i} + a\sin(s)\mathbf{j} + t\mathbf{k}$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq h$, as shown in Figure 11.15.

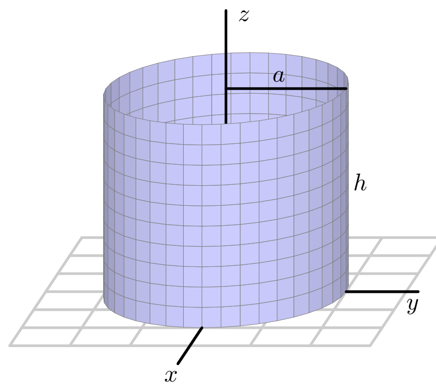

Figure 11.15: A cylinder.

(a) Set up an iterated integral to determine the surface area of this cylinder.

(b) Evaluate the iterated integral.

(c) Recall that one way to think about the surface area of a cylinder is to cut the cylinder horizontally and find the perimeter of the resulting cross sectional circle, then multiply by the height. Calculate the surface area of the given cylinder using this alternate approach, and compare your work in (b).

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Activity 11.19.

Let $z = f(x, y)$ define a smooth surface, and consider the corresponding parameterization $\mathbf{r}(s, t) = \langle s, t, f(s, t)\rangle$.

(a) Let D be a region in the domain of f. Using Equation **??**, show that the area, S, of the surface defined by the graph of f over D is

$$S = \iint_D \sqrt{(f_x(x, y))^2 + (f_y(x, y))^2 + 1}\, dA.$$

(b) Use the formula developed in (a) to calculate the area of the surface defined by $f(x, y) = \sqrt{4 - x^2}$ over the rectangle $D = [-2, 2] \times [0, 3]$.

(c) Observe that the surface of the solid describe in (b) is half of a circular cylinder. Use the standard formula for the surface area of a cylinder to calculate the surface area in a different way, and compare your result from (b).

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11.7 Triple Integrals

Preview Activity 11.7. Consider a solid piece granite in the shape of a box $B = \{(x, y, z) : 0 \leq x \leq 4, 0 \leq y \leq 6, 0 \leq z \leq 8\}$, whose density varies from point to point. Let $\delta(x, y, z)$ represent the mass density of the piece of granite at point (x, y, z) in kilograms per cubic meter (so we are measuring x, y, and z in meters). Our goal is to find the mass of this solid.

Recall that if the density was constant, we could find the mass by multiplying the density and volume; since the density varies from point to point, we will use the approach we did with two-variable lamina problems, and slice the solid into small pieces on which the density is roughly constant.

(a) Partition the interval $[0, 4]$ into 2 subintervals of equal length, the interval $[0, 6]$ into 3 subintervals of equal length, and the interval $[0, 8]$ into 2 subintervals of equal length. This partitions the box B into sub-boxes as shown in Figure 11.16.

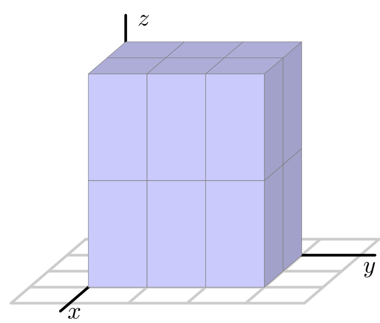

Figure 11.16: A partitioned three-dimensional domain.

(b) Let $0 = x_0 < x_1 < x_2 = 4$ be the endpoints of the subintervals of $[0, 4]$ after partitioning. Draw a picture of Figure 11.16 and label these endpoints on your drawing. Do likewise with $0 = y_0 < y_1 < y_2 < y_3 = 6$ and $0 = z_0 < z_1 < z_2 = 8$

What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 2? the length of Δy? of Δz?

(c) The partitions of the intervals $[0, 4]$, $[0, 6]$ and $[0, 8]$ partition the box B into sub-boxes. How many sub-boxes are there? What is volume ΔV of each sub-box?

(d) Let B_{ijk} denote the sub-box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Say that we choose a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in the i, j, kth sub-box for each possible combination of i, j, k. What is the meaning of $\delta(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$? What physical quantity will $\delta(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$ approximate?

(e) What final step(s) would it take to determine the exact mass of the piece of granite?

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Activity 11.20.

(a) Set up and evaluate the triple integral of $f(x, y, z) = x - y + 2z$ over the box $B = [-2, 3] \times [1, 4] \times [0, 2]$.

(b) Let S be the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$. A picture of S is shown at right in Figure 11.17. Our goal in what follows is to set up an iterated integral of the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} \int_{z=?}^{z=?} \delta(x, y, z)\, dz\, dy\, dx \tag{11.2}$$

to represent the mass of S in the setting where $\delta(x, y, z)$ tells us the density of S at the point (x, y, z). Our particular task is to find the limits on each of the three integrals.

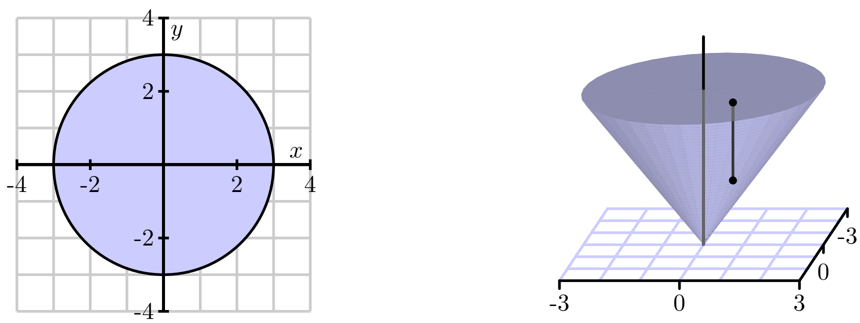

Figure 11.17: At right, the cone; at left, its projection.

i. If we think about slicing up the solid, we can consider slicing the domain of the solid's projection onto the xy-plane (just as we would slice a two-dimensional region in $\mathbb{R}^2$), and then slice in the z-direction as well. The projection of the solid is onto the xy-plane is shown at left in Figure 11.17. If we decide to first slice the domain of the solid's projection perpendicular to the x-axis, over what range of constant x-values would we have to slice?

ii. If we continue with slicing the domain, what are the limits on y on a typical slice? How do these depend on x? What, therefore, are the limits on the middle integral?

iii. Finally, now that we have thought about slicing up the two-dimensional domain that is the projection of the cone, what are the limits on z in the innermost integral? Note that over any point (x, y) in the plane, a vertical slice in the z direction will involve a range of values from the cone itself to its flat top. In particular, observe that at least one of these limits is not constant but depends on x and y.

iv. In conclusion, write an iterated integral of the form (11.2) that represents the mass of the cone S.

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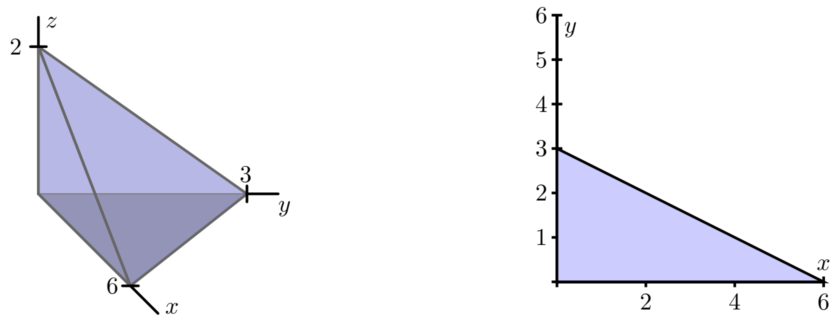

Figure 11.18: The tetrahedron and its projection.

Activity 11.21.

There are several other ways we could have set up the integral to give the mass of the tetrahedron in Example **??**.

(a) How many different iterated integrals could be set up that are equal to the integral in Equation (**??**)?

(b) Set up an iterated integral, integrating first with respect to z, then x, then y that is equivalent to the integral in Equation (**??**). Before you write down the integral, think about Figure 11.18, and draw an appropriate two-dimensional image of an important projection.

(c) Set up an iterated integral, integrating first with respect to y, then z, then x that is equivalent to the integral in Equation (**??**). As in (b), think carefully about the geometry first.

(d) Set up an iterated integral, integrating first with respect to x, then y, then z that is equivalent to the integral in Equation (**??**).

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Activity 11.22.

A solid S is bounded below by the square $z = 0$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ and above by the surface $z = 2 - x^2 - y^2$. A picture of the solid is shown in Figure 11.19.

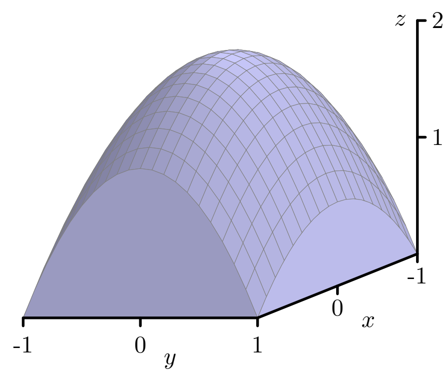

Figure 11.19: The solid bounded by the surface $z = 2 - x^2 - y^2$.

(a) Set up (but do not evaluate) an iterated integral to find the volume of the solid S.

(b) Set up (but do not evaluate) iterated integral expressions that will tell us the center of mass of S, if the density at point (x, y, z) is $\delta(x, y, z) = x^2 + 1$.

(c) Set up (but do not evaluate) an iterated integral to find the average density on S using the density function from part (b).

(d) Use technology appropriately to evaluate the iterated integrals you determined in (a), (b), and (c); does the location you determined for the center of mass make sense?

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11.8 Triple Integrals in Cylindrical and Spherical Coordinates

Preview Activity 11.8. In the following questions, we investigate the two new coordinate systems that are the subject of this section: cylindrical and spherical coordinates. Our goal is to consider some examples of how to convert from rectangular coordinates to each of these systems, and vice versa. Triangles and trigonometry prove to be particularly important.

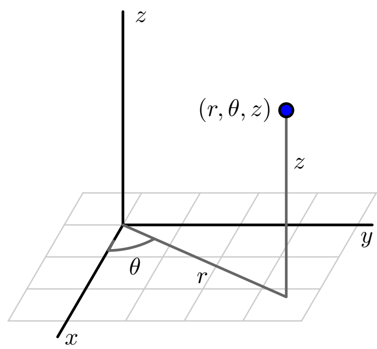

Figure 11.20: The cylindrical coordinates of a point.

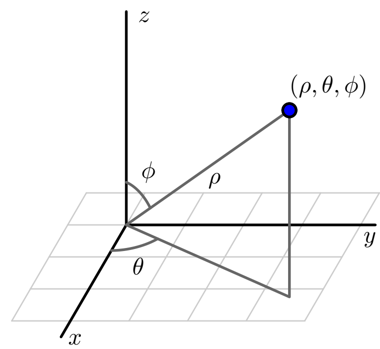

Figure 11.21: The spherical coordinates of a point.

The cylindrical coordinates of a point in $\mathbb{R}^3$ are given by (r, θ, z) where r and θ are the polar coordinates of the point (x, y) and z is the same z coordinate as in Cartesian coordinates. An illustration is given in Figure 11.20.

(a) Find cylindrical coordinates for the point whose Cartesian coordinates are $(-1, \sqrt{3}, 3)$. Draw a labeled picture illustrating all of the coordinates.

(b) Find the Cartesian coordinates of the point whose cylindrical coordinates are $\left(2, \frac{5\pi}{4}, 1\right)$. Draw a labeled picture illustrating all of the coordinates.

The spherical coordinates of a point in $\mathbb{R}^3$ are ρ (rho), θ, and ϕ (phi), where ρ is the distance from the point to the origin, θ has the same interpretation it does in polar coordinates, and ϕ is the angle between the positive z axis and the vector from the origin to the point, as illustrated in Figure 11.21.

For the following questions, consider the point P whose Cartesian coordinates are $(-2, 2, \sqrt{8})$.

(c) What is the distance from P to the origin? Your result is the value of ρ in the spherical coordinates of P.

(d) Determine the point that is the projection of P onto the xy-plane. Then, use this projection to find the value of θ in the polar coordinates of the projection of P that lies in the plane. Your result is also the value of θ for the spherical coordinates of the point.

(e) Based on the illustration in Figure 11.21, how is the angle ϕ determined by ρ and the z coordinate of P? Use a well-chosen right triangle to find the value of ϕ, which is the final component in the spherical coordinates of P. Draw a carefully labeled picture that clearly illustrates the values of ρ, θ, and ϕ in this example, along with the original rectangular coordinates of P.

(f) Based on your responses to (c), (d), and (e), if we are given the Cartesian coordinates (x, y, z) of a point Q, how are the values of ρ, θ, and ϕ in the spherical coordinates of Q determined by x, y, and z?

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Activity 11.23.

In this activity, we graph some surfaces using cylindrical coordinates. To improve your intuition and test your understanding, you should first think about what each graph should look like before you plot it using technology.[5]

(a) Plot the graph of the cylindrical equation $r = 2$, where we restrict the values of θ and z to the intervals $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. What familiar shape does the resulting surface take? How does this example suggest that we call these coordinates *cylindrical coordinates*?

(b) Plot the graph of the cylindrical equation $\theta = 2$, where we restrict the other variables to the values $0 \leq r \leq 2$ and $0 \leq z \leq 2$. What familiar surface results?

(c) Plot the graph of the cylindrical equation $z = 2$, using $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What does this surface look like?

(d) Plot the graph of the cylindrical equation $z = r$, where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What familiar surface results?

(e) Plot the graph of the cylindrical equation $z = \theta$ for $0 \leq \theta \leq 4\pi$. What does this surface look like?

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[5] e.g., http://www.math.uri.edu/~bkaskosz/flashmo/cylin/ – to plot $r = 2$, set r to 2, θ to s, and z to t – to plot $\theta = \pi/3$, set $\theta = \pi/3$, $r = s$, and $z = t$, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

Activity 11.24.

A picture of a cylindrical box, $B = \{(r, \theta, z) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2, z_1 \leq z \leq z_2\}$, is shown in Figure 11.22. Let $\Delta r = r_2 - r_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta z = z_2 - z_1$. We want to determine the volume ΔV of B in terms of Δr, $\Delta\theta$, Δz, r, θ, and z.

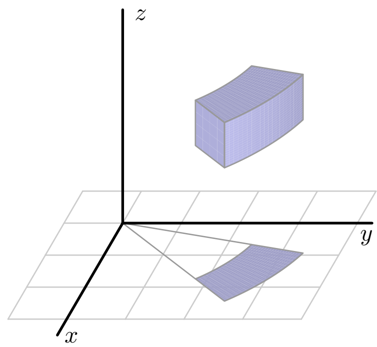

Figure 11.22: A cylindrical box.

(a) Appropriately label Δr, $\Delta\theta$, and Δz in Figure 11.22.

(b) Let ΔA be the area of the projection of the box, B, onto the xy-plane, which is shaded blue in Figure 11.22. Recall that we previously determined the area ΔA in polar coordinates in terms of r, Δr, and $\Delta\theta$. In light of the fact that we know ΔA and that z is the standard z coordinate from Cartesian coordinates, what is the volume ΔV in cylindrical coordinates?

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Activity 11.25.

In each of the following questions, set up, but do not evaluate, the requested integral expression.

(a) Let S be the solid bounded above by the graph of $z = x^2 + y^2$ and below by $z = 0$ on the unit circle. Determine an iterated integral expression in cylindrical coordinates that gives the volume of S.

(b) Suppose the density of the cone defined by $r = 1 - z$, with $z \geq 0$, is given by $\delta(r, \theta, z) = z$. A picture of the cone is shown in Figure 11.23, and the projection of the cone onto the xy-plane in given in Figure 11.24. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

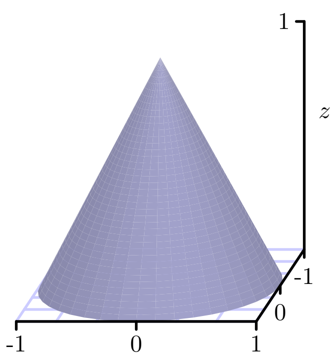

Figure 11.23: The cylindrical cone $r = 1 - z$.

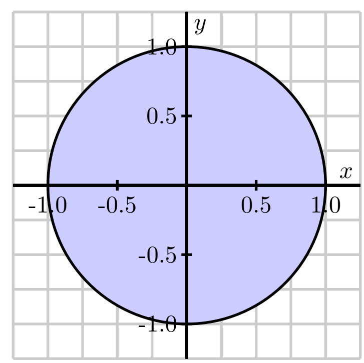

Figure 11.24: The projection into the xy-plane.

(c) Determine an iterated integral expression in cylindrical coordinates whose value is the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the cone $z = 4 - \sqrt{x^2 + y^2}$. A picture is shown in Figure 11.25.

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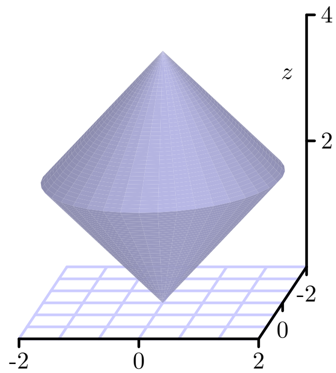

Figure 11.25: A solid bounded by the cones $z = \sqrt{x^2 + y^2}$ and $z = 4 - \sqrt{x^2 + y^2}$.

Activity 11.26.

In this activity, we graph some surfaces using spherical coordinates. To improve your intuition and test your understanding, you should first think about what each graph should look like before you plot it using technology.[6]

(a) Plot the graph of $\rho = 1$, where θ and ϕ are restricted to the intervals $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. What is the resulting surface? How does this particular example demonstrate the reason for the name of this coordinate system?

(b) Plot the graph of $\phi = \frac{\pi}{3}$, where ρ and θ are restricted to the intervals $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. What familiar surface results?

(c) Plot the graph of $\theta = \frac{\pi}{6}$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$. What familiar shape arises?

(d) Plot the graph of $\rho = \theta$, for $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. How does the resulting surface appear?

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[6]e.g., http://www.flashandmath.com/mathlets/multicalc/paramsphere/surf_graph_sphere.html – to plot $\rho = 2$, set ρ to 2, θ to s, and ϕ to t, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

Activity 11.27.

To find the volume element dV in spherical coordinates, we need to understand how to determine the volume of a spherical box of the form $\rho_1 \leq \rho \leq \rho_2$ (with $\Delta\rho = \rho_2 - \rho_1$), $\phi_1 \leq \phi \leq \phi_2$ (with $\Delta\phi = \phi_2 - \phi_1$), and $\theta_1 \leq \theta \leq \theta_2$ (with $\Delta\theta = \theta_2 - \theta_1$). An illustration of such a box is given in Figure 11.26. This spherical box is a bit more complicated than the cylindrical box we encountered earlier. In this situation, it is easier to approximate the volume ΔV than to compute it directly. Here we can approximate the volume ΔV of this spherical box with the volume of a Cartesian box whose sides have the lengths of the sides of this spherical box. In other words,

$$\Delta V \approx |PS|\,|\overset{\frown}{PR}|\,|\overset{\frown}{PQ}|,$$

where $|\overset{\frown}{PR}|$ denotes the length of the circular arc from P to R.

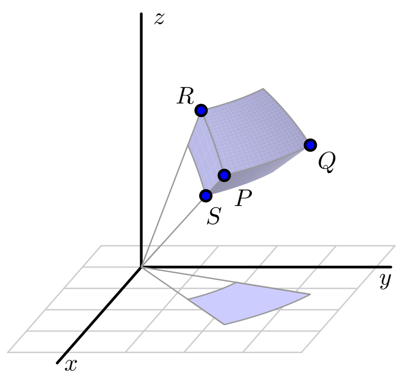

Figure 11.26: A spherical box.

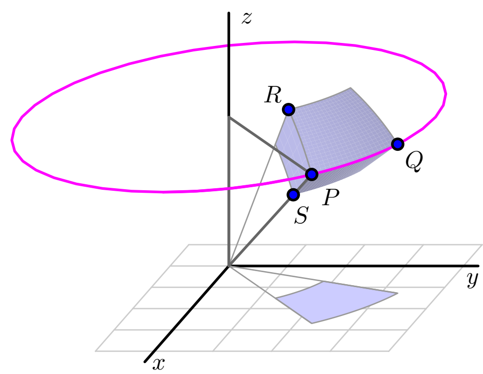

Figure 11.27: A spherical volume element.

(a) What is the length $|PS|$ in terms of ρ?

(b) What is the length of the arc $\overset{\frown}{PR}$? (Hint: The arc $\overset{\frown}{PR}$ is an arc of a circle of radius ρ_2, and arc length along a circle is the product of the angle measure (in radians) and the circle's radius.)

(c) What is the length of the arc $\overset{\frown}{PQ}$? (Hint: The arc $\overset{\frown}{PQ}$ lies on a horizontal circle as illustrated in Figure 11.27. What is the radius of this circle?)

(d) Use your work in (a), (b), and (c) to determine an approximation for ΔV in spherical coordinates.

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Activity 11.28.

We can use spherical coordinates to help us more easily understand some natural geometric objects.

(a) Recall that the sphere of radius a has spherical equation $\rho = a$. Set up and evaluate an iterated integral in spherical coordinates to determine the volume of a sphere of radius a.

(b) Set up, but do not evaluate, an iterated integral expression in spherical coordinates whose value is the mass of the solid obtained by removing the cone $\phi = \frac{\pi}{4}$ from the sphere $\rho = 2$ if the density δ at the point (x, y, z) is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. An illustration of the solid is shown in Figure 11.28.

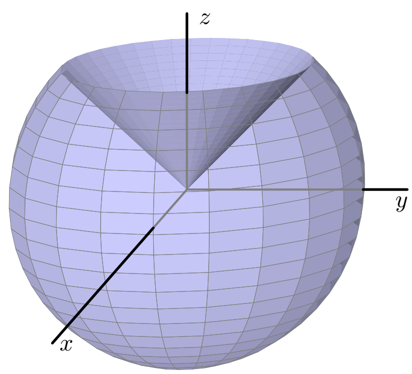

Figure 11.28: The solid cut from the sphere $\rho = 2$ by the cone $\phi = \frac{\pi}{4}$.

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11.9 Change of Variables

Preview Activity 11.9. Consider the double integral

$$I = \iint_D x^2 + y^2 \, dA, \tag{11.3}$$

where D is the upper half of the unit disk.

(a) Write the double integral I given in Equation (11.3) as an iterated integral in rectangular coordinates.

(b) Write the double integral I given in Equation (11.3) as an iterated integral in polar coordinates.

When we write the double integral (11.3) as an iterated integral in polar coordinates we make a change of variables, namely

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta). \tag{11.4}$$

We also then have to change dA to $r\,dr\,d\theta$. This process also identifies a "polar rectangle" $[r_1, r_2] \times [\theta_1, \theta_2]$ with the original Cartesian rectangle, under the transformation[7] in Equation (11.4). The vertices of the polar rectangle are transformed into the vertices of a closed and bounded region in rectangular coordinates.

To work with a numerical example, let's now consider the polar rectangle P given by $[1, 2] \times [\frac{\pi}{6}, \frac{\pi}{4}]$, so that $r_1 = 1$, $r_2 = 2$, $\theta_1 = \frac{\pi}{6}$, and $\theta_2 = \frac{\pi}{4}$.

(c) Use the transformation determined by the equations in (11.4) to find the rectangular vertices that correspond to the polar vertices in the polar rectangle P. In other words, by substituting appropriate values of r and θ into the two equations in (11.4), find the values of the corresponding x and y coordinates for the vertices of the polar rectangle P. Label the point that corresponds to the polar vertex (r_1, θ_1) as (x_1, y_1), the point corresponding to the polar vertex (r_2, θ_1) as (x_2, y_2), the point corresponding to the polar vertex (r_1, θ_2) as (x_3, y_3), and the point corresponding to the polar vertex (r_2, θ_2) as (x_4, y_4).

(d) Draw a picture of the figure in rectangular coordinates that has the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) as vertices. (Note carefully that because of the trigonometric functions in the transformation, this region will not look like a Cartesian rectangle.) What is the area of this region in rectangular coordinates? How does this area compare to the area of the original polar rectangle?

$\bowtie$

[7] A *transformation* is another name for function: here, the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ define a function $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$ so that T is a function (transformation) from $\mathbb{R}^2$ to $\mathbb{R}^2$. We view this transformation as mapping a version of the x-y plane where the axes are viewed as representing r and θ (the r-θ plane) to the familiar x-y plane.

Activity 11.29.

Consider the change of variables

$$x = s + 2t \quad \text{and} \quad y = 2s + \sqrt{t}.$$

Let's see what happens to the rectangle $T = [0,1] \times [1,4]$ in the st-plane under this change of variable.

(a) Draw a labeled picture of T in the st-plane.

(b) Find the image of the st-vertex $(0,1)$ in the xy-plane. Likewise, find the respective images of the other three vertices of the rectangle T: $(0,4)$, $(1,1)$, and $(1,4)$.

(c) In the xy-plane, draw a labeled picture of the image, T', of the original st-rectangle T. What appears to be the shape of the image, T'?

(d) To transform an integral with a change of variable, we need to determine the area element dA for image of the transformed rectangle. How would find the area of the xy-figure T'? (Hint: Remember what the cross product of two vectors tells us.)

◁

Activity 11.30.

Find the Jacobian when changing from rectangular to polar coordinates. That is, for the transformation given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, determine a simplified expression for the quantity

$$\left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right|.$$

What do you observe about your result? How is this connected to our earlier work with double integrals in polar coordinates?

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Activity 11.31.

Consider the problem of finding the area of the region D' defined by the ellipse $x^2 + \frac{y^2}{4} = 1$. Here we will make a change of variables so that the pre-image of the domain is a circle.

(a) Let $x(s,t) = s$ and $y(s,t) = 2t$. Explain why the pre-image of the original ellipse (which lies in the xy plane) is the circle $s^2 + t^2 = 1$ in the st-plane.

(b) Recall that the area of the ellipse D' is determined by the double integral $\iint_{D'} 1\, dA$. Explain why

$$\iint_{D'} 1\, dA = \iint_{D} 2\, ds\, dt$$

where D is the disk bounded by the circle $s^2 + t^2 = 1$. In particular, explain the source of the "2" in the st integral.

(c) Without evaluating any of the integrals present, explain why the area of the original elliptical region D' is 2π.

◁

Activity 11.32.

Let D' be the region in the xy-plane bounded by the lines $y = 0$, $x = 0$, and $x + y = 1$. We will evaluate the double integral

$$\iint_{D'} \sqrt{x+y}(x-y)^2 \, dA \tag{11.5}$$

with a change of variables.

(a) Sketch the region D' in the xy plane.

(b) We would like to make a substitution that makes the integrand easier to antidifferentiate. Let $s = x+y$ and $t = x-y$. Explain why this should make antidifferentiation easier by making the corresponding substitutions and writing the new integrand in terms of s and t.

(c) Solve the equations $s = x + y$ and $t = x - y$ for x and y. (Doing so determines the standard form of the transformation, since we will have x as a function of s and t, and y as a function of s and t.)

(d) To actually execute this change of variables, we need to know the st-region D that corresponds to the xy-region D'.

i. What st equation corresponds to the xy equation $x + y = 1$?
ii. What st equation corresponds to the xy equation $x = 0$?
iii. What st equation corresponds to the xy equation $y = 0$?
iv. Sketch the st region D that corresponds to the xy domain D'.

(e) Make the change of variables indicated by $s = x+y$ and $t = x-y$ in the double integral (11.5) and set up an iterated integral in st variables whose value is the original given double integral. Finally, evaluate the iterated integral.

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